METHODS OF GEOMETRY OF DIFFERENTIAL EQUATIONS IN ANALYSIS OF THE INTEGRABLE FIELD THEORY MODELS

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Abstract. In this paper, we investigate the algebraic and geometric properties of the hyperbolic Toda equations $u_{xy} = \exp(Ku)$ associated with nondegenerate symmetrizable matrices K. A hierarchy of analogs to the potential modified Korteweg–de Vries equation $u_t = u_{xxx} + u_x^3$ is constructed, and its relation with the hierarchy for the Korteweg–de Vries equation $T_t = T_{xxx} + TT_x$ is established. Group–theoretic structures for the dispersionless (2+1)-dimensional Toda equation $u_{xy} = \exp(-u_{zz})$ are obtained. Geometric properties of the multi–component nonlinear Schrödinger equation type systems $\Psi_t = \boldsymbol{i}\Psi_{xx} + \boldsymbol{i}f(|\Psi|)\,\Psi$ (multi–soliton complexes) are described.

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> ‘You boil it in sawdust: you salt it in glue:
> You condense it with locusts and tape:
> Still keeping one principal object in view —
> To preserve its symmetrical shape.’
>
> *The Hunting of the Snark*, Lewis Carroll.

The Toda equation ([93]) and the Toda type equations associated with the semisimple Lie algebras ([66]) play an important role in the models of modern conformal field theory. We recall that the Toda equations appear in gravity theory ([2, 11]), in the Yang–Mills field theory ([67]), in differential geometry ([29, 41]), and in the classification problems for nonlinear partial differential equations ([100]). Also, the Toda equations are related with the integrable dynamical systems ([22]), the Frobenius manifolds, and the associative algebra structures ([16]). The detailed study of the following systems that appear in the above–mentioned theories is reduced to the study of the Toda equations: these systems are the antiselfdual vacuum Einstein equations, the Yang–Mills equations, the Gauss–Mainardi–Codazzi equations for complex curves in the Kähler manifolds, the dynamical equations for the Laplace invariants of differential equations, the Korteweg–de Vries equation, and the WDVV (Witten–Dijkraaf–H. Verlinde–E. Verlinde) equation, respectively.

The algebraic approach towards the description of properties of the two–dimensional hyperbolic Toda equations

$$\boldsymbol{u}_{xy} = \exp(K\boldsymbol{u})$$

was developed in the papers by A. N. Leznov and M. V. Saveliev ([66]), V. G. Drinfel'd and V. V. Sokolov ([22]), B. A. Dubrovin ([16]) *et al.* In these papers, the Toda equations are interpreted as the flat connection equations on semisimple complex Lie algebras (or the Kac–Moody algebras) such that K is the corresponding Cartan matrix. Then the Toda equations are said to be associated with the Lie algebras (respectively, the Kac–Moody algebras). These equations are exactly integrable ([66]). In the fundamental paper [22], the integrable Drinfel'd–Sokolov hierarchies were assigned to the Toda equations associated with the Kac–Moody algebras. The Drinfel'd–Sokolov equations

are the bi–Hamiltonian Korteweg–de Vries type equations. Still, the Lie–algebraic approach does not provide the exhaustive description of geometric properties of the Toda equations themselves. Indeed, the structure of generators of the Noether symmetries Lie subalgebra for the Toda equation, the existence of recursion operators for its symmetry algebra, or the relation between conservation laws for the Toda equation and the Hamiltonian structures for the Korteweg–de Vries equation was not revealed. In particular, until recently it remained unnoticed that all these properties of the Toda equation are preserved for the general case of the Toda equations $\boldsymbol{u}_{xy} = \exp(K\boldsymbol{u})$ associated with nondegenerate symmetrizable matrices K, which are not assumed to be the Cartan matrices.

The homological methods developed by I. S.Krasil'shchik, V. V. Lychagin, A. M. Vinogradov ([10, 62, 63, 94]), and their scientific school are a powerful tool in the analysis of algebraic and geometric structures for differential equations. Owing to essential achievements of this theory, it was quite natural to apply the homological methods to the study of the Toda equation and related systems.

In this article, we carry out the detailed investigation of the geometric properties of the hyperbolic Toda equations. Also, we construct new Hamiltonian evolution systems associated with the Toda equations and establish their nontrivial links with other important classes of the mathematical physics equations such as the Korteweg–de Vries equation.

Introduction

The discrete one–dimensional Toda lattice ([93])

$$\ddot{q}_n = \exp(q_{n-1} - q_n) - \exp(q_n - q_{n+1}), \qquad n \in \mathbb{Z}, \quad q_n = q_n(\tau) \quad (1)$$

is a model nonlinear integrable equation in mathematical physics. The equation

$$\varepsilon^2 \cdot q_{\tau\tau} = \exp\bigl(q(z-\varepsilon) - q(z)\bigr) - \exp\bigl(q(z) - q(z+\varepsilon)\bigr) \qquad (2)$$

is a continuous analog of Eq. (1). Here $\varepsilon \geq 0$ and $q = q(\tau, z)$. Equation (2) is then obtained by using the relation $q_n(\tau) = q(\tau, n\varepsilon)$. The equation

$$u_{\tau\tau} = \pm D_z^2 \circ \exp(u), \qquad u = u(\tau, z),$$

is the limit for the family of equations (2) as $\varepsilon \to +0$. Here D_z is the total derivative with respect to z and $u_n(\tau) \equiv q_{n-1} - q_n = u(\tau, n\varepsilon)$. In the paper [16] Dubrovin proved that Eq. (2) is reduced to the nonlinear Schrödinger equation if $\varepsilon = \boldsymbol{i}$. In this paper, we consider the multi–component analogs (multi–soliton complexes, see [1])

$$\Psi_t = \boldsymbol{i}\Psi_{xx} + \boldsymbol{i} f(|\Psi|)\,\Psi, \qquad (3)$$

of the nonlinear Schrödinger equation. Here Ψ is the m-component vector, $\boldsymbol{i} = \sqrt{-1}$, and $f \in C^\infty(\mathbb{R})$.

The passage from one–dimensional equation (1) to the two–dimensional Toda equations

$$u_{xy} = \exp(Ku) \qquad (4)$$

was described in the paper [66]. By [66], the acceleration $\partial^2/\partial\tau^2$ with respect to the time $t \in \mathbb{R}$ is replaced by the d'Alambertian $\partial^2/\partial x\partial y$; we also assume $(x,\, y) \in \mathbb{C}^2$. The Toda fields $u^j(x,\, y)$ are counted by the superscript $j \in [1,\, r]$. Suppose $\langle\vec{\alpha}_i\rangle$ is the rood system of a semisimple complex Lie algebra $\mathfrak{g}$ of rank r and

$$K = \left\| k_{ij} = \frac{2(\vec{\alpha}_i, \vec{\alpha}_j)}{|\vec{\alpha}_j|^2} \right\|$$

is its Cartan matrix. In [66], equations (4) were supposed to be associated with the Cartan matrices by default. In the sequel, we extend the existing picture and analyse the properties of the Toda equations (4) such that K is a nondegenerate symmetrizable matrix (see precise definitions below).

Let the matrix K in Eq. (4) be the Cartan matrix of the Lie algebra of type A_r. Then we set

$$u^j(x,y) = u(x,y,z)\bigr|_{z=j\varepsilon}$$

for each $1 \leq j \leq r$. The continuous limit of Eq. (4) as $r \to \infty$ and $\varepsilon \to +0$ is the dispersionless Toda equation

$$u_{xy} = \exp(-u_{zz}). \qquad (5)$$

This equation appeares in many problems of mathematical physics, for example, in the gravity theory ([11], see also the paper [87] and references therein).

In this paper, we investigate algebraic and geometric properties of the Toda equations (4–5) and the nonlinear Schrödinger equation (3). Also, we describe the relation between the Toda equation (4) and the hierarchies of the Korteweg–de Vries equations (9) and (16) (see below). We apply modern cohomological methods and algorights in our study of the geometric properties of all these equations. By using the algebraic approach, we reject the idea of a decisive break–through and the inevitably immense calculations in coordinates. We operate with the notions of homological algebra in the category of infinite prolongations of differential equations instead.

The results exposed in this article are also found in the sequence of publications [44]–[60].

The paper is organized as follows. In the introduction, we fix notation, formulate necessary definitions, and give a brief list of the final results.

In Chapter 1, we describe important properties of the symmetry algebra as well as conservation laws for the Toda equations. In Sec. 1, we review some known properties of the hyperbolic Toda equations associated with the semisimple Lie algebras. Also, we define the Toda equations associated with nondegenerate symmetrizable matrices. In what follows, we study these equations that require less restrictions upon the matrix K. In Sec. 2 within Chapter 1, we obtain the Noether symmetries of the Lagrangian for the Toda equations. In Sec. 3, we construct a continuum of recursion operators for the Toda equation.

In Chapter 2, we construct the commutative hierarchy $\mathfrak{A}$ of analogs for the potential modified Kortweg–de Vries equation. This hierarchy is identified with a commutative subalgebra of the Noether symmetries algebra for the Toda equations. Also, some aspects of the Hamiltonian formalism for the hyperbolic Toda equations themselves are discussed. Then we relate the symmetry hierarchy $\mathfrak{A}$ with the higher Korteweg–de Vries equations.

In Chapter 3, we illustrate the methods of geometry of partial differential equations. Namely, we investigate the properties of the dispersionless Toda equation and the multi–component analog of the nonlinear Schrödinger equation, which is related with the former equation.

In Chapter 4, we consider Bäcklund transformations for the Toda equation associated with the algebra $\mathfrak{sl}_2(\mathbb{C})$ (the Liouville equation) and related systems. Also, we obtain one–parametric families of these transformations. Examples of integrating Bäcklund transformations in nonlocal variables are given in Chapter 4, zero–curvature representations are pointed out, and the relations between all these structures are analysed.

In Appendix A, several geometrical methods of solving the boundary problems for equations of the mathematical physics are listed.

Basic definitions and notation. First let us introduce some notions from the geometry of partial differential equations. We follow [4, 5, 10, 62, 63, 94] and the articles [45, 47].

Differential equations and their symmetries. Let $\mathcal{E} = \{F^\alpha(\boldsymbol{x},\, \boldsymbol{u},\, \mathbf{p}) = 0,\ 1 \le \alpha \le r\}$ be a partial differential equation of order k imposed by the smooth functions F^α on n independent variables $\vec{x} = {}^t(x^1,\, \ldots,\, x^n)$, r dependent variables $\boldsymbol{u} = {}^t(u^1,\, \ldots,\, u^r)$, and the derivatives

$$\mathbf{p} = \{p^j_\sigma \mid p^j_\sigma = \partial^{|\sigma|}u^j/\partial(x^1)^{i_1}\ldots\partial(x^n)^{i_n},\\ \sigma = \{i_1, \ldots, i_n\}, \quad |\sigma| = i_1 + \cdots + i_n \le k\}.$$

Consider the trivial r-dimensional smooth fibre bundle

$$\pi\colon \mathbb{R}^r \times \mathbb{R}^n \to \mathbb{R}^n.$$

The space $J^k(\pi)$ of kth jets of the bundle π is the union $\bigcup_x J^k_x$, where J^k_x is the set of all equivalence classes $[s]^k_x$ of sections s in the bundle π such that the sections s are tangent with order k at the point $x \in \mathbb{R}^n$. Define the sequence of smooth fibre bundles

$$\pi_{k+1,k}\colon J^{k+1}(\pi) \to J^k(\pi)$$

by the formula

$$\pi_{k+1,k}\bigl([s]^{k+1}_x\bigr) = [s]^k_x.$$

We denote by π_k the smooth fibre bundle $\pi_k\colon J^k(\pi) \to \mathbb{R}^n$ defined by $\pi_k\bigl([s]^k_x\bigr) = x$. Also, by $\mathcal{F}_k(\pi)$ we denote the ring of smooth (C^∞) functions on $J^k(\pi)$. Finally, we set $\mathcal{F}_{-\infty}(\pi) = C^\infty(\mathbb{R}^n)$.

We treat the variables $\boldsymbol{x}$, $\boldsymbol{u}$, and $\mathbf{p}$ as coordinates in the jet space $J^k(\pi)$. By the Hadamars lemma ([38]), two sections s, $s' \in \Gamma(\pi)$ are equivalent in $J^k_x(\pi)$ iff their partial derivatives at the point x coincide up to order k.

A submanifold

$$\mathcal{E} = \{F^\alpha = 0\} \subset J^k(\pi)$$

is a *differential equation* of order not greater than k imposed on r functions of n independent variables. An equation $\mathcal{E} \subset J^k(\pi)$ is *regular* if the mapping

$$\pi_k\big|_{\mathcal{E}}\colon \mathcal{E} \to M = \mathbb{R}^n$$

is a surjection.

Definition 1. Consider the sections s of the fiber bundle π such that $[s]^k_x = \theta$ for some point $\theta \in J^k(\pi)$. Consider the tangent planes to the graphs Γ^k_s of the kth jets of s. The linear span of all these tangent planes is the *Cartan plane* $C_\theta = C^k_\theta$ at the point θ. The union $\mathcal{C}$ of the mappings $\theta \mapsto C_\theta$ by all $\theta \in J^k(\pi)$ is the *Cartan distribution.*

Definition 2. The subset

$$\mathcal{E}^{(l)} = \{\theta_{k+l} = [s]_x^{k+l}, s \in \Gamma(\pi) \mid \\ \mid j_k(s)(x) \text{ is tangent to } \mathcal{E} \text{ at the point } \theta_k = [s]_x^k \text{ with order } \geq l\}$$

in $J^{k+l}(\pi)$ is the lth *prolongation* of an equation $\mathcal{E} \subset J^k(\pi)$. The inverse limit

$$\mathcal{E}^\infty = \operatorname{proj} \lim_{l \to \infty} \mathcal{E}^{(l)}$$

with respect to the projections

$$\pi_{l+1,l} \colon J^{l+1}(\pi) \to J^l(\pi)$$

is called the *infinite prolongation* of the equation $\mathcal{E}$. In what follows, we omit the superscript ∞ if it is clear from the context that the infinite prolongation $\mathcal{E}^\infty$ but not the differential equation $\mathcal{E}$ is considered.

The infinite prolongation of the empty equation $\{0 = 0\} \simeq J^0(\pi)$ is called the *infinite jets space* $J^\infty(\pi)$. The projections

$$\pi_{\infty,k} \colon J^\infty(\pi) \to J^k(\pi)$$

and

$$\pi_\infty \colon J^\infty(\pi) \to \mathbb{R}^n$$

are defined by the formulas

$$\pi_{\infty,k}(\theta_\infty) = \theta_k$$

and

$$\pi_\infty(\theta_\infty) = x,$$

respectively, where

$$\theta_\infty = \{x, \theta_k \in J^k(\pi) \mid k \in \mathbb{N}\} \in J^\infty(\pi).$$

The algebra $\mathcal{F}(\pi)$ of smooth functions on $J^\infty(\pi)$ is defined by using the following procedure. We set

$$\mathcal{F}(\pi) = \bigcup_k \mathcal{F}_k(\pi), \quad k \in \{-\infty\} \cup \mathbb{N}.$$

The module $\Lambda^i(\pi)$ of differential i-forms on $J^\infty(\pi)$ is defined by the relation

$$\Lambda^i(\pi) = \bigoplus_{k \geq 0} \Lambda^i(J^k(\pi)).$$

The restriction $\mathcal{C}_\mathcal{E}$ of the Cartan distribution $\mathcal{C}$ onto the equation $\mathcal{E}^\infty$ is an n-dimensional Frobenius distribution. This distribution defines the decomposition of the tangent space to $\mathcal{E}^\infty$ into the horizontal and vertial subspaces. The horizontal component is generated by the restrictions of the total derivatives

$$D_i = \widehat{\frac{\partial}{\partial x^i}}$$

onto $\mathcal{E}^\infty$. These restrictions are denoted by $\bar{D}_i$. The dual description of the Cartan distribution $\mathcal{C}_\mathcal{E}$ in the language of differential forms is also useful. The de Rham differential on $\mathcal{E}^\infty$ is representable as the restriction onto $\mathcal{E}^\infty$ of the sum composed by the horizontal differential

$$\mathrm{d}_h = \sum_{i=1}^{n} \mathrm{d}x^i \otimes D_i$$

(this is the lifting of the differential on the base of the fibre bundle π) and the Cartan differential

$$\mathrm{d}_\mathcal{C} = \mathrm{d} - \mathrm{d}_h.$$

Therefore the space $\Lambda^l(\mathcal{E})$ of differential l-forms on the equation $\mathcal{E}^\infty$ is the direct sum

$$\Lambda^l(\mathcal{E}) = \bigoplus_{i+j=l} \bar{\Lambda}^i(\mathcal{E}) \otimes \mathcal{C}^j\Lambda(\mathcal{E})$$

of the horizontal i-forms $\bar{\Lambda}^i(\mathcal{E})$ and the Cartan j-forms $\mathcal{C}^j\Lambda(\mathcal{E})$. The Cartan forms $\omega^j_\sigma \equiv \mathrm{d}_\mathcal{C}(u^j_\sigma)$ constitute a basis in $\mathcal{C}^1\Lambda(\mathcal{E})$. Here u^j_σ are coordinates on $\mathcal{E}^\infty$.

The horizontal differential $\bar{\mathrm{d}}_h$ generates the horizontal de Rham complex

$$0 \to \mathcal{F}(\pi) \xrightarrow{\mathrm{d}_h} \bar{\Lambda}^1(\pi) \xrightarrow{\mathrm{d}_h} \ldots \xrightarrow{\mathrm{d}_h} \bar{\Lambda}^n(\pi) \to 0$$

of the space $J^\infty(\pi)$. The cohomologies of this complex are called the horizontal cohomologies and are denoted by $\bar{H}^i(\pi)$. We denote by $\bar{H}^i(\mathcal{E})$ the cohomologies of the restriction of this complex onto the equation $\mathcal{E}^\infty$. From the definition it follows that the equivalence classes $[\eta] \in \bar{H}^{n-1}(\mathcal{E})$ are conservation laws for the equation $\mathcal{E}$.

Definition 3. (1) The *evolutionary derivation* is the operator

$$\partial_\varphi = \sum_{j,\sigma} D_\sigma(\varphi^j)\, \partial/\partial p^j_\sigma,$$

where $\varphi^j \in C^\infty(J^k(\pi))$ for some k and D_σ is the composition of total derivatives D_i that corresponds to the multiindex σ. We denote by $\varkappa$ the $\mathcal{F}(\pi)$-module $\Gamma(\pi) \otimes_{\mathcal{F}_{-\infty}} \mathcal{F}(\pi)$. Also, we denote

$$\hat{\varkappa} = \mathrm{Hom}_{\mathcal{F}(\pi)}(\varkappa, \bar{\Lambda}^n(\pi)).$$

(2) Suppose Δ_ψ is the nonlinear differential operator determined by a function $\psi \in C^\infty(J^k(\pi))$. The operator ℓ_ψ that acts by the rule

$$\ell_\psi(\varphi) = \partial_\varphi(\psi)$$

is called the *universal linearization* operator of Δ_ψ. In coordinates, we have

$$\ell_\psi = \left\| \sum_\sigma \frac{\partial \psi^i}{\partial u^j_\sigma} \cdot D_\sigma \cdot \mathbf{1}_{ij} \right\|.$$

Any Lie field X that preserves the Cartan distribution $\mathcal{C}$ can be decomposed to the sum $X = \mathrm{Э}_\varphi + Y$, where $Y \in \mathcal{C}$ and $\mathrm{Э}_\varphi$ is the evolutionary vector field. Any infinitesimal transformation of the space $J^0(\pi)$ can be extended up to the Lie field. In coordinates, the lifting rules are the following. The field

$$\hat{X} = \sum_i a_i\, D_i + \sum_{i,j} \mathrm{Э}_{b_j - a_i p_i^j}$$

is assigned to the field

$$X^0 = \sum_i a_i \frac{\partial}{\partial x^i} + \sum_j b_j \frac{\partial}{\partial u^j}.$$

An example is found in Eq. (35) on page 26.

Definition 4. A *symmetry* of an equation $\mathcal{E}^\infty$ is a π-vertical vector field X such that X preserves the Cartan distribution $\mathcal{C}_\mathcal{E} = \bigcup_{\theta\in\mathcal{E}^\infty} C_\theta$:

$$[X, \mathcal{C}_\mathcal{E}] \subset \mathcal{C}_\mathcal{E}.$$

Theorem 1 ([10]). *Suppose $\mathcal{E} \subset J^k(\pi)$ is an equation*

$$\{F^1 = 0, \ldots, F^r = 0\},$$

such that

$$\pi_{\infty,0}(\mathcal{E}^\infty) = J^0(\pi).$$

Then the symmetry Lie algebra $\operatorname{sym}\mathcal{E}^\infty$ *is isomorphic to the Lie algebra of solutions to the system of the determining equations*

$$\mathrm{Э}_\varphi(F) = 0 \quad \textit{on } \mathcal{E}$$

or, equivalently, of the equations

$$\ell_F(\varphi) = 0 \quad \textit{on } \mathcal{E}$$

owing to the definition of the linearization operator ℓ. Here $\varphi \in \varkappa\big|_{\mathcal{E}^\infty}$. The bracket

$$\{\varphi, \psi\}_{\mathcal{E}^\infty} = \left(\mathrm{Э}_\varphi(\psi) - \mathrm{Э}_\psi(\varphi)\right)\big|_{\mathcal{E}^\infty}.$$

endowes the algebra $\operatorname{sym}\mathcal{E}^\infty$ *of solutions φ with the Lie algebra structure.*

In what follows, we identify the generating sections

$$\varphi = {}^t(\varphi^1, \ldots, \varphi^r)$$

of symmetries

$$\mathrm{Э}_\varphi = \sum_{i,\sigma} \bar{D}_\sigma(\varphi^i) \cdot \frac{\partial}{\partial u^i_\sigma}$$

(we denote $u^i_\sigma \equiv D_\sigma(u^i)$) of differential equations with the symmetries $\mathrm{Э}_\varphi$ themselves (see [22, 49]). Roughly speaking, the component φ^i of a generating section φ measure the velocity u^i_t of the dependent variable u^i along the "integral trajectories" of the field $\mathrm{Э}_\varphi$.

Definition 5. Suppose $\varphi(u,\ \dots,\ u_\sigma)$ is a symmetry of a differential equation $\mathcal{E}$, where σ is a multiindex and

$$u_\sigma = \partial^{|\sigma|}u/\partial(x^1)^{\sigma_1}\dots\partial(x^n)^{\sigma_n}$$

is the derivative of the dependent variable u. Assume that the symmetry φ has the flow

$$A_\tau\colon u(x,0)\mapsto u(x,\tau).$$

This flow is defined on solutions of the equation $u_\tau = \varphi$ and maps the solutions $s(x) = u(x,\tau)|_{\tau=0}$ of equation $\mathcal{E}$ to solutions of the same equation at points $\tau > 0$. A solution $s(x)$ of the equation $\mathcal{E}$ is called *φ-invariant* if $s(x)$ is a stationary solution of the evolution equation

$$u_\tau = \varphi(u,\dots,u_\sigma).$$

Therefore, the search of the φ-invariant solutions for a given equation $\mathcal{E} = \{F = 0\}$ is reduced to solving the system $\{F = 0,\ \varphi = 0\}$.

No we specify an important class of *Hamiltonian* equations within the set of evolution equations $\boldsymbol{u}_t = f(t,\ \boldsymbol{x},\ \boldsymbol{u},\ \mathbf{p})$. Here we formulate the definitions of the Poisson bracket and the Hamiltonian operators class (see [10, 26, 42] for further detailes).

Definition 6. Let $A \in \mathcal{C}\mathrm{Diff}(\hat{\varkappa}(\pi),\varkappa(\pi))$ be an $(r\times r)$-matrix operator in total derivatives:

$$A = \|A^{ij}\|, \qquad A^{ij} = A^{ij}_\sigma\cdot D_\sigma,$$

and let $\mathcal{L}_1,\ \mathcal{L}_2 \in \bar{H}^n(\pi)$ be two Lagrangians. By definition, set the Poisson bracket (the variational bracket) on $\bar{H}^n(\pi)$ by the formula

$$\{\mathcal{L}_1,\mathcal{L}_2\}_A = \big\langle \mathbf{E}(\mathcal{L}_1), A(\mathbf{E}(\mathcal{L}_2))\big\rangle = \Big[\sum_{i,j}\frac{\delta\mathcal{L}_1}{\delta u^i}\cdot A^{ij}\left(\frac{\delta\mathcal{L}_2}{\delta u^j}\right)\mathrm{d}\boldsymbol{x}\Big], \tag{6}$$

where $\langle\cdot,\cdot\rangle$ is the natural coupling $\varkappa(\pi)\times\hat{\varkappa}(\pi)\to\bar{H}^n(\pi)$ and $[\cdot]$ denotes the equivalence class of differential forms.

Definition 7. Suppose the operator A is subject to the assumptions of the previous definition. Then the operator A is called *Hamiltonian* if the Poisson bracket defined in Eq. (6) endows $\bar{H}^n(\pi)$ with the Lie algebra structure over $\mathbb{R}$ such that the followin relations hold

$$\{\mathcal{L}_1,\mathcal{L}_2\}_A + \{\mathcal{L}_2,\mathcal{L}_1\}_A = 0, \tag{7a}$$

$$\{\{\mathcal{L}_1,\mathcal{L}_2\}_A,\mathcal{L}_3\}_A + \{\{\mathcal{L}_2,\mathcal{L}_3\}_A,\mathcal{L}_1\}_A + \{\{\mathcal{L}_3,\mathcal{L}_1\}_A,\mathcal{L}_2\}_A = 0. \tag{7b}$$

The bracket $\{\cdot,\cdot\}_A$ is the *Hamiltonian structure.*

Two Hamiltionian operators A_1, A_2 are *compatible* if $\lambda A_1 + \mu A_2$ is a Hamiltonian operator again for any $\lambda,\ \mu\in\mathbb{R}$.

Condition (7a) is satisfied iff $A+A^*=0$. Refer [10, 63, 62, 42] for the suitable criteria[1] which check whether Jacobi's identity (7b) holds for a given operator A. For example, any skew–symmetric $\mathcal{C}$-differential operator A with constant coefficients is Hamiltonian.

Definition 8. The evolution equation

$$\boldsymbol{u}_t = A(\mathbf{E}_{\boldsymbol{u}}(\mathcal{H})) \tag{8}$$

is a *Hamiltonian evolution equation* assigned to the Hamiltonian $\mathcal{H} \in \bar{H}^n(\pi)$ by the Hamiltonian operator A.

Example 1 ([70]). The Korteweg–de Vries equation

$$T_t = -\beta\, T_{xxx} + 3T\, T_x, \qquad \beta = \text{const}, \tag{9}$$

is hamiltonian with respect to the pair

$$\hat{B}_1 = D_x, \qquad \hat{B}_2 = -\beta\, D_x^3 + 2T \cdot D_x + T_x \tag{10}$$

of the compatible Hamiltonian operators. Indeed, we have

$$T_t = \hat{B}_1 \circ \mathbf{E}_T\bigl((\tfrac{1}{2}\beta T_x^2 + \tfrac{1}{2}T^3)\,\mathrm{d}x\bigr) = \hat{B}_2 \circ \mathbf{E}_T\bigl(\tfrac{1}{2}T^2\,\mathrm{d}x\bigr).$$

Conservation laws. In this subsection, we formulate important definitions and statements concerning conservation laws for differential equations. Also, we recall the relation between symmetries and conservation laws for the Euler equations.

Definition 9. A *conservation law*

$$[\eta] \in \bar{H}^{n-1}(\mathcal{E}) \equiv \{\omega \in \bar{\Lambda}^{n-1}(\mathcal{E}) \mid \bar{\mathrm{d}}_h(\omega) = 0\}/ \\ / \{\omega \in \bar{\Lambda}^{n-1}(\mathcal{E}) \,|\, \omega = \bar{\mathrm{d}}_h\gamma,\ \gamma \in \bar{\Lambda}^{n-2}(\mathcal{E})\}$$

for an equation $\mathcal{E}$ is an equivalence class of horizontal $(n-1)$-forms $\eta \in \bar{\Lambda}^{n-1}(\mathcal{E})$ closed on $\mathcal{E}$,

$$\bar{\mathrm{d}}_h\eta = \nabla(F)\,\mathrm{d}\boldsymbol{x}.$$

Here

$$\bar{\mathrm{d}}_h = \sum_{i=1}^{n} \mathrm{d}x^i \otimes \bar{D}_i$$

is the restriction of the horizontal differential d_h on $\mathcal{E}$, D_i is the total derivative with respect to x^i, ∇ is an operator in total derivatives, and the equivalence is the factorization by exact forms $\bar{\mathrm{d}}_h\gamma$, $\gamma \in \bar{\Lambda}^{n-2}(\mathcal{E})$. Representatives η of the equivalence classes $[\eta] \in \bar{H}^{n-1}(\mathcal{E})$ are called *conserved currents* for the equation $\mathcal{E}$.

[1] The of notion of the *Poisson bi-vectors* is a useful intrument in the Hamiltonian operators theory (see [26, 42]). For example, the first Hamiltonian structure for the Korteweg–de Vries equation (9) is $\mathbf{1} \wedge D_x$ and the second structure is $\mathbf{1} \wedge \hat{B}_2$, see Eq. (10). A bi-vector A is Poisson if and only if A satisfies the equation $[\![A, A]\!] = 0$, where $[\![\cdot,\cdot]\!]$ is the Schouten bracket. A pait of the Poisson bi-vectors A_1 and A_2 is compatible if $[\![A_1, A_2]\!] = 0$. In the papers [52, 53], the equation $[\![A, A]\!] = 0$ was considered in a more general situation such that A is not necessarily a bi-vector.

Example 2 ([50]). The horizontal 2-form

$$\eta = u_{xz}\exp(-u_{zz})\,\mathrm{d}x\wedge\mathrm{d}y + \bigl(\tfrac{1}{2}u_{xz}^2 - u_{xx}\bigr)\,\mathrm{d}x\wedge\mathrm{d}z$$

is a conserved current for the dispersionless Toda equation

$$u_{xy} = \exp(-u_{zz}).$$

Indeed, the relation

$$\bar{D}_y\bigl(\tfrac{1}{2}u_{xz}^2 - u_{xx}\bigr) = \bar{D}_z\bigl(u_{xz}\,\exp(-u_{zz})\bigr) \tag{11}$$

holds on that equation.

Definition 10. A regular equation $\mathcal{E} = \{F = 0\}$ is called *ℓ-normal* if the condition

$$\nabla\circ\bar{\ell}_F = 0$$

implies $\nabla = 0$.

Proposition 2 ([10])**.** *Let n be the number of independent variables x^1, ..., x^n and let $\mathcal{E}$ be a regular equation. Consider a coordinate neighbouhood $\Omega(\theta^\infty)\subset\mathcal{E}^\infty$ of a point $\theta^\infty\in\mathcal{E}^\infty$ and suppose that there is a set $\{v\}$of the internal coordinates v on Ω such that the total derivatives $D_i(v)$ can be expressed in terms of these coordinates $\{v\}$ for any i, $1\le i<n$. Then the equation $\mathcal{E}$ is ℓ-normal.*

If $\mathcal{E}$ is an ℓ-normal equation, then the compatibility complex ([10, 63]) for the equation $\mathcal{E}$ has length 2, and the equation satisfies the assumptions of the '2-line' theorem.

Remark 1. The Maxwell, the Yang–Mills, and the Einstein equations are *not* ℓ-normal since there is a nontrivial dependence between the equations that is provided by the gauge symmetry pseudogroup.

Now we need a test that checks whether a given equation is ℓ-normal. We have

Example 3 ([10, 63]). The evolution equations are ℓ-normal.

Conservation laws $[\eta]$ for the ℓ-normal equations are described by their *generating sections* $\vec{\psi}\equiv\nabla^*(1)\in\hat{\varkappa}$. We see that

$$\mathrm{d}_h\eta = \langle\nabla(F),1\rangle = \langle F,\nabla^*(1)\rangle + \mathrm{d}_h\gamma.$$

Then the coupling

$$\langle\nabla^*(1),F\rangle = \mathrm{d}_h(\eta-\gamma) \tag{12}$$

is an exact horizontal n-form. Suppose η is trivial and therefore $\nabla = 0$. Then, obviously, then $\vec{\psi} = 0$.

Lemma 3 ([94])**.** *Let $\mathcal{E} = \{F = 0\}$ be an ℓ-normal equation and assume that $H^n(\mathcal{E})\subset\bar{H}^n(\mathcal{E})$ (e.g., $H^n(\mathcal{E}) = 0$). Suppose that the generating section $\vec{\psi}$ of a current η is zero. Then the conserved current η is trivial.*

In what follows, we shall use the following remarkable theorem many times.

Theorem 4 ([94])**.** *Let $\mathcal{E} = \{F = 0\}$ be an ℓ-normal equation in the fibre bundle $\pi\colon \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$. Then the generating sections $\vec{\psi}$ of conservation laws $[\eta]$ satisfy the equation*

$$\bar{\ell}_F^*(\vec{\psi}_\eta) = 0, \tag{13}$$

where ℓ_F^* is the operator formally adjoint to ℓ_F and $\bar{\ell}_F^*$ is the restriction of ℓ_F^* onto the equation $\mathcal{E}$.

Proof. Apply the Euler operator $\mathbf{E}$ to both sides of Eq. (12). Thence we obtain

$$0 = \mathbf{E}(\langle\psi, F\rangle) = \ell^*_{\langle\psi,F\rangle}(1) = \ell_F^*(\psi) + \ell_\psi^*(F) = 0 \tag{14}$$

by the Leibnitz rule. Now restrict Eq. (14) onto the prolongation of $F = 0$ and obtain the determining equation (13) imposed on the generating sections and satisfied on the initial equation $\mathcal{E}$. □

The generating sections ψ_η are often named *gradients* of the conservation laws $[\eta]$ (see [22]) by the following reason. The generating sections for evolution equations lie in the image of the Euler operator (that is, the "gradient" $\delta/\delta u$) applied to the corresponding conserved density.

Lemma 5 ([63, 96])**.** *Let $\mathcal{E} = \{u_t = f(t, x, u, u_1, \ldots)\}$ be an evolution equation. Assume that*

$$\eta = \eta_0\, d\boldsymbol{x} + \sum_{i=1}^{n} (-1)^{i-1} \eta_i\, dt \wedge dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n$$

is a conserved current for $\mathcal{E}$: $\bar{d}_h(\eta) = 0$, that is,

$$\bar{D}_t(\eta_0) + \sum_i \bar{D}_{x^i}(\eta_i) = 0.$$

Then its generating function is

$$\psi_\eta = \mathbf{E}(\eta_0) \equiv \ell^*_{\eta_0}(1).$$

The proof of Lemma 5 is straightforward.

The Euler–Lagrange equation $\mathcal{E}_{E-L}$ assigned to a Lagrangian $\mathcal{L} \in \bar{H}^n(\pi)$ is

$$\mathcal{E}_{E-L} = \{G \equiv \mathbf{E}_{\boldsymbol{u}}(\mathcal{L}) = 0\}. \tag{15}$$

We recall that the Helmholz condition $\ell_G = \ell_G^*$ is valid for the image G of the Euler operator $\mathbf{E}$.

The correlation between conservation laws $[\eta]$, their generating sections ψ_η, and the Noether symmetries $\varphi_{\mathcal{L}} \in \operatorname{sym} \mathcal{L}$ of an Euler–Lagrange equation $\mathcal{E} = \{\mathbf{E}(\mathcal{L} = 0\}$ is guided by the following version of the Noether theorem.

Theorem 6 ([6])**.** *Let $\mathcal{E} = \{\mathbf{E}(\mathcal{L}) = 0\}$ be the Euler-Lagrange equation for a Lagrangian $\mathcal{L}$. Then the evolutionary derivation $\mathfrak{D}_\varphi$ is a Noether symmetry of the Lagrangian $\mathcal{L}$:*

$$\mathfrak{D}_\varphi(\mathcal{L}) = 0$$

if and only if φ is the generating section of a conservation law $[\eta]$: $\mathrm{d}_h\eta = 0$ on $\mathcal{E}$.

Proof. Suppose φ is the generating section of a conservation laws $[\eta]$ for the equation $\mathcal{E}$:

$$\mathrm{d}_h(\eta) = \langle 1, \Box(F)\rangle = \langle \Box^*(1), F\rangle + \mathrm{d}_h(\gamma),$$

where the coupling $\langle\cdot,\cdot\rangle$ takes values in the space $\bar{\Lambda}^n(\pi)$ of horizontal n-forms. Thence, $\langle\varphi, F\rangle = \mathrm{d}_h(\eta-\gamma)$ is an exact horizontal n-form. We have $F = \mathbf{E}(\mathcal{L}) = \ell_L^*(1)$ by the initial assumption, whence we easily obtain

$$\langle\varphi, F\rangle = \langle\varphi, \ell_L^*(1)\rangle = \langle\ell_L(\varphi), 1\rangle + \mathrm{d}_h\beta = \langle\mathfrak{D}_\varphi(L), 1\rangle + \mathrm{d}_h\beta.$$

Therefore, φ is a Noether symmetry. The second implication in the theorem's statement, the sufficiency, is proved by inverting the reasonings. □

Coverings. Let $\mathcal{E}$ be a differential equation in the fibre bundle $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $\mathcal{E}^\infty$ be the infinite prolongation of $\mathcal{E}$. The n-dimensional Cartan plane $\mathcal{C}_\theta \subset T_\theta(\mathcal{E}^\infty)$ is defined at each point $\theta \in \mathcal{E}^\infty$. The Cartan distribution $\mathcal{C}_\mathcal{E}$ on $\mathcal{E}^\infty$ is Frobenius:

$$[\mathcal{C}_\mathcal{E}, \mathcal{C}_\mathcal{E}] \subset \mathcal{C}_\mathcal{E}.$$

In local coordinates, $\mathcal{C}_\mathcal{E}$ is defined by n vector fields $\bar{D}_1, \ldots, \bar{D}_n$, where $\bar{D}_i$ is the restriction of the total derivative with respect to the ith independent variable x^i onto $\mathcal{E}^\infty$.

Definition 11 ([10])**.** The equation $\tilde{\mathcal{E}}^\infty$ with the n-dimensional Cartan distribution $\tilde{\mathcal{C}}$ and the regular map τ are called a *covering* over the equation $\mathcal{E}^\infty$ if at any point $\theta \in \tilde{\mathcal{E}}^\infty$ the tangent map $\tau_{*,\theta}$ is an isomorphim of the plane $\tilde{\mathcal{C}}_\theta$ onto the Cartan plane $\mathcal{C}_{\tau(\theta)}$ on the equation $\mathcal{E}^\infty$ at the point $\tau(\theta)$. The equation $\tilde{\mathcal{E}}$ is called the *covering equation*. The dimension of the fibre in the bundle τ is the *dimension of the covering*.

In coordinates, the structure of a covering is realized in the following way. Locally, we can realize the manifold $\tilde{\mathcal{E}}$ and the mapping $\tau\colon \tilde{\mathcal{E}} \to \mathcal{E}^\infty$ as the direct product $\mathcal{E}^\infty \times W$ ($W \subseteq \mathbb{R}^N$ is an open subset, $0 < N \leq \infty$) and the natural projection $\mathcal{E}^\infty \times W \to \mathcal{E}^\infty$, respectively. Then we describe the distribution $\tilde{\mathcal{C}}$ on $\tilde{\mathcal{E}}$ by the vector fields

$$\tilde{D}_i = \bar{D}_i + \sum\nolimits_{j=1}^N X_{ij}\,\frac{\partial}{\partial s^j}, \qquad i = 1, \ldots, n,$$

where $X_{ij} \in C^\infty(\tilde{\mathcal{E}})$ are the coefficients of the τ-vertical fields on $\tilde{\mathcal{E}}$ and s_1, ..., and s_N are the Cartesian coordinates in $\mathbb{R}^N$. Then, the Frobenius condition $[\tilde{\mathcal{C}}, \tilde{\mathcal{C}}] \subset \tilde{\mathcal{C}}$ of integrability for the distribution $\tilde{\mathcal{C}}$ is equivalent to the system of relations

$$[\tilde{D}_i, \tilde{D}_j] = 0, \qquad i, j = 1, \ldots, n,$$

which are equivalent to the equations

$$\tilde{D}_i(X_{jk}) = \tilde{D}_j(X_{ik})$$

that hold for all $i, j = 1, \ldots, n$, $0 \leq k \leq N$.

The coordinates s_i are called nonlocal variables. In coordinates x^i, u^j_σ, and s_j the rules $\tilde{D}_i(s_j) = X_{ij}$ to differentiate the nonlocal variables s_j as well as the initial equation $\mathcal{E}^\infty$ define the covering equation $\tilde{\mathcal{E}}$.

Example 4. Again, consider the Korteweg–de Vries equation (9). Extend the set of variables t, x, $T_j \equiv D_x^j(T)$ with the "nonlocality" $s = \int T\,\mathrm{d}x$:

$$s_x = T, \qquad s_t = -\beta\, s_{xxx} + \tfrac{3}{2} s_x^2. \tag{16}$$

We see that the equation that covers (9) is the potential Korteweg–de Vries equation.

A symmetry of the covering equation $\tilde{\mathcal{E}}$ is a *nonlocal symmetry* of the equation $\mathcal{E}^\infty$. Suppose that the field $\hat{X}$ is a symmetry of an equation $\mathcal{E}^\infty$ and $\tau\colon \tilde{\mathcal{E}} \to \mathcal{E}^\infty$ is a covering. Two radically different cases are possible:

(1) the symmetry $\hat{X}$ of the equation $\mathcal{E}^\infty$ can be extended up to a symmetry $\tilde{X}$ of the covering equation $\tilde{\mathcal{E}}$, and
(2) the converse: any lifting of the symmetry $\hat{X}$ is not a symmetry of the covering equation.

In the second case, the field $\hat{X}$ generates the one-parametric family of equations $\tilde{\mathcal{E}}_t$ that cover the equation $\mathcal{E}^\infty$ for any t.

Recursion operators. In this subsection, we describe an explicit method ([62]) of constructing recursion operators for differential equations. The Cartan generating forms approach developed by I. S. Krasil'shchik is the key-point of our reasonings.

Let $\mathcal{E} = \{F = 0\}$ be a determined system of differential equations, that is, the number m of dependent variables in this equation is equal to the number r of equations. Suppose $\varphi \in \operatorname{sym} \mathcal{E}^\infty$ is a symmetry of this equation and consider an r-component column

$$\omega = {}^t(\omega^1, \ldots, \omega^r)$$

such that its components

$$\omega^i \in C^\infty(\mathcal{E}^\infty) \otimes \mathcal{C}^1\Lambda(\mathcal{E})$$

are the Cartan 1-forms that vanish on the Cartan distribution $\mathcal{C}_\mathcal{E}$ on $\mathcal{E}^\infty$ and the coefficients at these forms are functions on $\mathcal{E}^\infty$. Obviously,

the component-wise action $\ni_\varphi \rfloor \omega \equiv \varphi'$ is again an element of the module $\varkappa$ of evolutionary derivations. The condition for φ' to be a symmetry of the initial equation $\mathcal{E}$ is

$$\bar{\ell}_F^{[1]}(\omega) = 0, \tag{17}$$

where $\bar{\ell}_F^{[1]}$ is the restriction of the linearization operator onto $\mathcal{C}^1\Lambda(\mathcal{E})$.

Now we formulate the rule that assigns differential operators R in total derivatives to the columns (the *generating forms*) ω. Suppose the components ω^i of a generating form $\omega = {}^t(\omega^1, \ldots, \omega^r)$ are

$$\omega^i = \sum_{j,\sigma} a_\sigma^{ij}\, \omega_\sigma^j.$$

Then we obtain

$$\ni_\varphi \rfloor \omega = {}^t\big(\ldots, \sum\nolimits_{j,\sigma} a_\sigma^{ij}\, D_\sigma(\varphi^j), \ldots\big).$$

We see that the component-wise action of the Cartan forms ω^i is equivalent to the action of the operators

$$\sum_{j,\sigma} a_\sigma^{ij}\, \ell_{u_\sigma^j}$$

on the components of the initial generating section $\varphi = {}^t(\varphi^1, \ldots, \varphi^r)$. Hence we deduce that the formal differential recursion operator R is the $(r \times r)$-matrix,

$$R = \left\| \sum\nolimits_\sigma a_\sigma^{ij}\, D_\sigma \right\|.$$

Still, we observe that "usually" equation (17) admits only the trivial solution

$$\omega_\varnothing = {}^t(\omega_\varnothing^1, \ldots, \omega_\varnothing^r)$$

that corresponds to the identity recursion operator

$$\mathrm{id}\colon \varphi \mapsto \varphi' = \varphi.$$

The explanation is that the recursion operators for known equations of mathematical physics usually contain summands that involve the negative powers $D_{x^\alpha}^{-1}$ of the total derivatives D_σ.

Example 5. The recursion operator $R_{\text{KdV}} = \hat{B}_2 \circ \hat{B}_1^{-1}$ for the Korteweg–de Vries equation (9) is equal to

$$R_{\text{KdV}} = -\beta\, D_x^2 + 2T + T_x \cdot D_x^{-1}. \tag{18a}$$

The recursion operator R_{pKdV} for the potential Korteweg–de Vries equation (16) is

$$R_{\text{pKdV}} = -\beta\, D_x^2 + 2s_x - D_x^{-1} \circ s_{xx}. \tag{18b}$$

From the geometric viewpoint, the situation is like follows. We obtain nontrivial solutions of Eq. (17) by extending the set of local coordinates $\langle x^1, \ldots, x^n, u^j_\sigma\rangle$ by the "nonlocal" variables s^i. These nonlocalities are defined by the compatible rules of their derivation with respect to the independent variables. Also, to each s^i we assign the Cartan form

$$\mathrm{d}_{\mathcal{C}}(s^i) = \mathrm{d}s^i - \sum_{j=1}^{n} s^i_{x^j}\,\mathrm{d}x^j,$$

which now belongs to some larger distribution $\mathcal{C}^1\Lambda(\tilde{\mathcal{E}})$.

Consider the restriction $\tilde{\ell}_F^{[1]}$ of the linearization operator ℓ_F onto the extended set of the Cartan forms. Substitute the new linearization operator in equation (17) and assume that a nontrivial solution ω appeared. Now we see that the generating form ω is coupled with some prolongation

$$\tilde{\ni}_{\varphi,A(\varphi)} = \ni_\varphi + \sum_i A_i(\varphi)\cdot\frac{\partial}{\partial s^i} + \cdots$$

of the initial evolutionary field $\ni_\varphi$. Possibly, this prolongation involves a larger (ingeneral, an infinite) set of nonlocal variables, see the paper [43]. The result

$$\varphi' = \tilde{\ni}_{\varphi,A(\varphi)} \lrcorner\, \omega$$

of the action of ω satisfies the equation

$$\tilde{\ell}_F(\varphi) = 0.$$

These sections $\varphi'(x, u^j_\sigma, s^i, \ldots)$ are called *shadows* of nonlocal symmetries. In general, there are shadows that cannot be extended up to a true nonlocal symmetry by using a prescribed set of the nonlocal variables s^i. An example is found in Proposition 58 on page 99.

Example 6. Consider the following one–dimensional covering over the potential Korteweg–de Vries equation (16). Introduce the nonlocal variable ζ such that

$$\zeta_x = -\tfrac{1}{2}s_1^2, \qquad \zeta_t = \beta s_1 s_3 - \tfrac{1}{2}\beta s_2^2 - s_1^3, \tag{19}$$

where $s_j \equiv D_x^j(s)$ and we similarly denote $\zeta_j \equiv \tilde{D}_x^j(\zeta)$. One easily checks that the covering equation is the Krichever–Novikov type equation

$$\zeta_t = -\beta\,\zeta_3 + \tfrac{3}{4}\beta\zeta_2^2\zeta_1^{-1} - 2\sqrt{2}(-\zeta_1)^{3/2}. \tag{20}$$

Then the equation

$$\tilde{\ell}^{[1]}_{\mathrm{pKdV}}(\omega) = 0$$

admits a nontrivial solution in the extended set of variables t, x, s_j, and ζ. Indeed, the Cartan generating form of the recursion operator for Eq. (16) is

$$\omega_{\mathrm{pKdV}} = -\beta\,\mathrm{d}_{\mathcal{C}}(s_2) + s_1\,\mathrm{d}_{\mathcal{C}}(s) - \mathrm{d}_{\mathcal{C}}(\zeta). \tag{21}$$

Now we describe the relation between recursion operators R in total derivatives, which are convenient in applications, and solutions ω solutions of the determining equation

$$\tilde{\ell}_F^{[1]}(\omega) = 0.$$

It is sufficient to solve this problem for the Cartan forms $\mathrm{d}_{\mathcal{C}}(s^i)$. Now it is clear that the problem is reduced to calculation of the linearization ℓ_{s^i} with respect to the nonlocal variable s^i. To do this, we use the following lemma.

Lemma 7. *Let $s \in \mathcal{F}(\pi)$ be a function; fix a superscript $i \in [1,\, n]$ of the independent variable x^i. Then the relation*

$$\ell_{D_i(s)} = D_i \circ \ell_s \tag{22}$$

holds.

Proof. Suppose $\varphi \in \varkappa$, then we obtain

$$\ell_{D_i(s)}(\varphi) = \text{Э}_\varphi \circ D_i(s) = D_i \circ \text{Э}_\varphi(s) = D_i \circ \ell_s(\varphi),$$

whence follows relation (22). □

Lemma 7 defines the rule

$$\tilde{\ell}_{s^j} = \tilde{D}_i^{-1} \circ \tilde{\ell}_{\tilde{D}_i(s^j)} \tag{23}$$

of calculating the linearization of a nonlocality s^j. Here $i \in [1,\, n]$ is arbitrary. Therefore, to each generating form $\omega = {}^t(\omega^1,\, \ldots,\, \omega^r)$, where

$$\omega^i = \sum_{j,\sigma} a_\sigma^{ij}\, \omega_\sigma^j + \sum_j a^{ij}\, \mathrm{d}_{\mathcal{C}}(s^j),$$

we assign the matrix operator

$$R = \left\| \sum_{j,\sigma} a_\sigma^{ij} \cdot D_\sigma \cdot \ell_{u^j} + \sum_j a^{ij} \cdot D_{x^k}^{-1} \circ \ell_{D_{x^k}(s^j)} \right\|,$$

where the index i enumerates the rows and $1 \leq k \leq n$ is arbitrary.

Example 7. Recursion operator (18b) corresponds to the generating Cartan form (21). Operator (18a) corresponds to the Cartan form

$$\omega_{\mathrm{KdV}} = -\beta\, \mathrm{d}_{\mathcal{C}}(T_2) + 2T\, \mathrm{d}_{\mathcal{C}}(T) + T_1\, \mathrm{d}_{\mathcal{C}}(s).$$

Bäcklund transformations. The notion of a covering is very useful in description of Bäcklund transformations between differential equations.

Definition 12 ([10])**.** Let $\mathcal{E}_i \subset J^{k_i}(\pi_i)$, $i = 1, 2$, be two differential equations and $\tau_i \colon \tilde{\mathcal{E}} \to \mathcal{E}_i^\infty$ be coverings with the same total space $\tilde{\mathcal{E}}$. Then the diagram

$$\mathcal{B}(\tilde{\mathcal{E}}, \tau_i, \mathcal{E}_i) = \left\{ \mathcal{E}_1 \xleftarrow{\tau_1} \tilde{\mathcal{E}} \xrightarrow{\tau_2} \mathcal{E}_2 \right\} \tag{24}$$

is called a *Bäcklund transformation* $\mathcal{B}(\tilde{\mathcal{E}}, \tau_i, \mathcal{E}_i)$ between the equations $\mathcal{E}_i$. Diagram (24) is called a *Bäcklund autotransformation* if $\mathcal{E}_1^\infty = \mathcal{E}_2^\infty = \mathcal{E}^\infty$.

In what follows, any Bäcklund transformation between equations $\mathcal{E}$ and $\mathcal{E}'$ is a system of differential relations imposed on the unknown functions u and u' such that the following property holds. Suppose the function u is a solution to the equation $\mathcal{E}$ and both functions u and u' satisfy these relations, then the function u' is a solution to the equation $\mathcal{E}'$, and *vice versa*.

Example 8. Bäcklund autotransformation for the two–dimensional Laplace equation $\Delta_2 v = 0$ is defined by the relations

$$v'_y = v''_x,\ v'_x = -v''_y.$$

Remark 2. Let $\tau_j\colon \tilde{\mathcal{E}}_j \to \mathcal{E}_j^\infty$, $j = 1, 2$, be two coverings and $\mu\colon \tilde{\mathcal{E}}_1 \to \tilde{\mathcal{E}}_2$ be a diffeomorphism that maps the Cartan distribution $\mathcal{C}_{\tau_1}\mathrm{D}(\tilde{\mathcal{E}}_1)$ into $\mathcal{C}_{\tau_2}\mathrm{D}(\tilde{\mathcal{E}}_2)$. Then the diagram $\mathcal{B}(\tilde{\mathcal{E}}_1,\ \tau_1,\ \tau_2\circ\mu,\ \mathcal{E}_j)$ is also a Bäcklund transformation between the equations $\mathcal{E}_j$ and these coverings τ_1 and $\tau_2\circ\mu$ are called *equivalent*.

Remark 3. Let $\tau\colon \tilde{\mathcal{E}} \to \mathcal{E}^\infty$ be a covering and $\mu\colon \tilde{\mathcal{E}} \to \tilde{\mathcal{E}}$ be a nontrivial diffeomorphism of manifolds that preserves the Cartan distribution, *e.g.*, a discrete symmetry that cannot be restricted onto $\mathcal{E}^\infty$. Then the diagram

$$\mathcal{E} \xleftarrow{\ \tau\ } \tilde{\mathcal{E}} \xrightarrow{\ \mu\ } \tilde{\mathcal{E}} \xrightarrow{\ \tau\ } \mathcal{E} \tag{25}$$

is also Bäcklund autotransformation for $\mathcal{E}$. In Chapter 4, we apply this contruction to the study of Bäcklund autotransformation for the Liouville equation $u_{xy} = \exp(2u)$.

Now we give a brief exposition of the mail results obtained in the present article.

Main results. In Chapter 1, we consider a problem standard in the geometry of differential equations. Namely, we describe relations between symmetries, conservation laws, the Noether symmetries, and recursion operators for the Toda equations. The structures obtained in this chapter are essentially used in what follows.

Let $K = \|k_{ij},\ 1 \le i,\ j \le r\|$ be a nondegenerate $(r\times r)$-matrix and $K^{-1} = \|k^{ij}\|$ be its inverse. Suppose there is the set $\vec{a}$ of real numbers $\{a_i \ne 0,\ 1 \le i \le r\}$ such that the symmetry condition $\kappa_{ij} = \kappa_{ji}$ holds fot the matrix $\kappa = \|\kappa_{ij}\|$ whose elements are

$$\kappa_{ij} = a_i \cdot k_{ij}.$$

Then we say that the matrix K is *symmetrizable*.

The hyperbolic Toda equations associated with a nondegenerate symmetrizable $(r\times r)$-matrix K are

$$\mathcal{E}_{\mathrm{Toda}} = \Bigl\{F_i \equiv u^i_{xy} - \exp\bigl(\sum_{j=1}^{r} k_{ij}u^j\bigr) = 0, \quad 1 \le i \le r\Bigr\}. \tag{26}$$

In particular, suppose $\mathfrak{g}$ is a semisimple Lie algebra of rank r, let $\{\alpha_i, 1 \leq i \leq r\}$ be the system of its simple roots, and $K = \|k_{ij} = 2(\alpha_i, \alpha_j) \cdot |\alpha_j|^{-2}, 1 \leq i, j \leq r\|$ be the Cartan matrix of the algebra $\mathfrak{g}$. Tne we set $a_i = |\alpha_i|^{-2}$. The Toda equation (26) associated with this matrix K is said to be ([66]) *associated with the Lie algebra* $\mathfrak{g}$.

The Toda equations (26) are the Euler–Lagrange equations with respect to the action functional

$$\mathcal{L}_{\text{Toda}} = \int L_{\text{Toda}} \, \mathrm{d}x \wedge \mathrm{d}y$$

with the density

$$L_{\text{Toda}} = -\tfrac{1}{2} \sum_{i,j=1}^{r} \kappa_{ij} u^i_x u^j_y - \sum_{i=1}^{r} a_i \cdot \exp\Bigl(\sum_{j=1}^{r} k_{ij} u^j\Bigr).$$

The canonical Hamiltonian structure is known for the Toda equations (see [77]).

Suppose K is a nondegenerate symmetrizable matrix. Then the Toda equation (26) associated with K admits at least one integral ([100]), that is, an expession dependent of u^j_σ whose total derivative $\bar{D}_y$ vanishes on this equation. Namely, we have

$$T = \tfrac{1}{2} \sum_{i,j=1}^{r} \kappa_{ij} u^i_x u^j_x - \sum_{i=1}^{r} a_i \cdot u^i_{xx} \in \ker \bar{D}_y. \tag{27}$$

By definition, put

$$T_j \equiv \bar{D}^j_x(T).$$

The differential consequences T_j to the functional T generate the subspace $\mathbf{T} \subset \ker \bar{D}_y$ within the kernel of the total derivative $\bar{D}_y$. Indeed, any smooth function Q supplies the functional

$$Q(x, \mathbf{T}) \equiv Q(x, T, T_1, \ldots, T_\mu) \in \ker \bar{D}_y.$$

We say that the nondegenerate symmetrizable matrix K is *generic* if the integral T in Eq. (27) is a unique solution to the equation $\bar{D}_y(T) = 0$ on the corresponding Toda equation (26).

Still, by a special choice of the matrix K one can achieve the situation such that the functional T will *not* be a unique integral for the Toda equation (26). By [90], a necessary and sufficient condition for r nontrivial independent solutions Ω^i of the equation

$$\bar{D}_y(\Omega^i) = 0$$

to exist is that K be the Cartan matrix of a semisimple Lie algebra $\mathfrak{g}$. The Toda equations associated with $\mathfrak{g}$ are exactly integrable ([65, 77]). From now on, by $\mathbf{\Omega} \subseteq \ker \bar{D}_y$ we denote the subspace in $\ker \bar{D}_y$, which is differentially generated by *all* solutions Ω^i to the equation $\bar{D}_y(\Omega^i) = 0$. The total number of these solutions is denoted by q: $1 \leq i \leq q \leq r$. We always assume $\Omega^1 \equiv T$.

By $\vec{\Delta} = |\Delta^i|$, where $\Delta^i = \sum\limits_{j=1}^{r} k^{ij}$, we denote the vector of the conformal dimensions of the Toda fields $\exp(u) \equiv {}^t(\exp(u^1), \ldots, \exp(u^r))$. The transformation

$$\begin{aligned} x &\mapsto \mathcal{X}(x), \\ y &\mapsto \mathcal{Y}(y), \\ u^i(x,y) &\mapsto \tilde{u}^i = u^i(\mathcal{X}, \mathcal{Y}) + \Delta^i \log \mathcal{X}'(x)\mathcal{Y}'(y) \end{aligned} \tag{28}$$

is a finite conformal symmetry of the Toda equation $\mathcal{E}_{\text{Toda}}$. The generating section φ of the infinitesimal form of conformal transformation (28) is

$$\varphi = \square(\phi(x)),$$

where the vector–valued, first–order differential operator $\square$ is

$$\square = u_x + \Delta \cdot \bar{D}_x \tag{29}$$

and ϕ is an arbitrary smooth function. The structure of the Lie algebra $\operatorname{sym} \mathcal{E}_{\text{Toda}}$ generators is described in the paper [75]. We have

(1) Suppose K is a generic nondegenerate $(r \times r)$-matrix and the operator $\square$ is assigned to this matrix by using formula (29). Then any symmetry of the Toda equation (26) is

$$\varphi = \square(\phi(x, \mathbf{\Omega})), \tag{30}$$

where ϕ is an arbitrary smooth function that depends on a set of the integrals $\mathbf{\Omega} \subset \ker \bar{D}_y$.

(2) Suppose the matrix K is subject to additional constraints such that the Toda equation (26) admits q independent integrals $\Omega^i \in \ker \bar{D}_y$, where $1 \le i \le q \le r$ and

$$\mathbf{\Omega} = \{\Omega^i_j \equiv \bar{D}^j_x(\Omega^i)\} \supset \mathbf{T}.$$

We also assume that the first–order operator $\square$ is the $(r \times q)$-matrix and satisfies some additional restrictions (see [75]). Then symmetries φ of the Toda equation are of the form (30) with respect to the new notation.

The Jacobi bracket on the symmetries $\varphi \in \operatorname{sym} \mathcal{E}_{\text{Toda}}$ induces a bracket on the arguments of the operator $\square$. Suppose

$$\varphi_1 = \square\,(\phi_1(x, \mathbf{T}))$$

and

$$\varphi_2 = \square\,(\phi_2(x, \mathbf{T})).$$

Then we have

$$\{\varphi_1, \varphi_2\} = \square\,(\phi_{\{1,2\}}),$$

where

$$\phi_{\{1,2\}} = \mathbf{\ni}_{\varphi_1}(\phi_2) - \mathbf{\ni}_{\varphi_2}(\phi_1) + \bar{D}_x(\phi_1)\,\phi_2 - \phi_1\,\bar{D}_x(\phi_2).$$

Here $\phi_{\{1,2\}} = \phi_{\{1,2\}}(x,\,\mathbf{T})$, since the evolution $\dot{T}_\phi$ of integral (27) along any symmetry $\varphi = \square\,(\phi)$ is equal to

$$\dot{T}_\phi = -\beta\,\bar{D}_x^3(\phi) + T\,\bar{D}_x(\phi) + \bar{D}_x(T\cdot\phi). \tag{31}$$

Here and below we use the notation

$$\beta \equiv \sum_{i=1}^{r} a_i \cdot \Delta^i.$$

Consider the operator applied to the function ϕ in the right-hand side of Eq. (31). We see that this operator is the second Hamiltonian structure $\hat{B}_2$ for the Korteweg–de Vries equation (9).

In this paper, we prove that the generating sections ψ_η of conservation laws $[\eta]$ and the Noether symmetries $\varphi_{\mathcal{L}}$ for the Toda equation (26) are related by the equation $\psi_\eta = \kappa\,\varphi_{\mathcal{L}}$. Hence we obtain

Theorem (theorem 18 on page 41)**.** (1) *Suppose* Ω^i *is an integral for the Toda equation* (55) *such that* $D_y(\Omega^i) = \tilde{\nabla}_i(F)$*. Then there is the operator* ∇_i *such that*

$$\tilde{\nabla}_i = \nabla_i \circ \square^* \circ \kappa.$$

In particular, consider the integral $\Omega^1 = T$ *defined in Eq.* (58)*. Then*

$$\nabla_1 = \mathbf{1}.$$

(2) *The Noether symmetries of the Toda equation are*

$$\varphi_{\mathcal{L}} = \square \circ \sum_i \nabla_i^* \circ \mathbf{E}_{\Omega^i}(Q(x,\bar{\mathbf{\Omega}})),$$

where $\Omega^i \in \ker \bar{D}_y$ *are the integrals for the equation* $\mathcal{E}_{\text{Toda}}$,

$$\mathbf{E}_{\Omega^i} = \sum_{j\geq 0} (-1)^j D_x^j \cdot \frac{\partial}{\partial \Omega^i_j}$$

is the Euler operator with respect to Ω^i, $\bar{\mathbf{\Omega}}$ *is an arbitrary set of differential consequences to* Ω^i*, and* Q *is a smooth function.*

Example 9 ([57]). The Noether symmetries $\varphi_{\mathcal{L}}$ of the Toda equation (55) associated with a generic nondegenerate symmetrizable matrix K are

$$\varphi_{\mathcal{L}} = \square \circ \mathbf{E}_T(Q(x,\mathbf{T}))$$

up to the transformation $x \leftrightarrow y$.

In Sec. 3, we construct a continuum of recursion operators for the symmetry algebra of the Toda equations. These operators are parameterized by arbitrary smooth functions. Although the structure of the symmetry algebra itself is known, see Eq. (30), presence of the recursion operators gives us additional information about the Toda equation and permits to establish the relation between $\mathcal{E}_{\text{Toda}}$ and other mathematical physics equations.

Theorem (theorem 24 on page 45). (1) *Equation* (55) *admits continuum of* local *recursion operators*

$$R\colon \operatorname{sym}\mathcal{E}_{\text{Toda}} \to \operatorname{sym}\mathcal{E}_{\text{Toda}},$$

which are

$$R = \square \circ \sum_{i,j} f_{ij}(x,\mathbf{\Omega}) \cdot \bar{D}_x^j \circ \ell_{\Omega^i}.$$

Here f_{ij} are arbitrary smooth functions and the linearizations ℓ_{Ω^i} with respect to the integrals Ω^i for the Toda equation are

$$\ell_{\Omega^i} = \Bigl(\dots, \underbrace{\textstyle\sum_\sigma \partial\Omega^i/\partial u^k_\sigma \cdot \bar{D}_\sigma,}_{k\text{th } \textit{component}} \dots\Bigr). \tag{32}$$

(2) *There is a continuum of* nonlocal *recursion operators for Eq.* (26)*, which are constructed in the following way. Assign the nonlocal variables s^i to the integrals Ω^i by the compatible differentiation rules*

$$s^i_x = \Omega^i, \quad s^i_y = 0.$$

The linearizations ℓ_{s^i} are defined by the formulas

$$\ell_{s^i} = \bar{D}_x^{-1} \circ \ell_{\Omega^i},$$

where ℓ_{Ω^i} are calculated by Eq. (32)*. Then the required recursion operators are*

$$R = \square \circ \sum_i f_i(x,\mathbf{s},\mathbf{\Omega}) \cdot \bar{D}_x^{-1} \circ \ell_{\Omega^i},$$

where the functions f_i are arbitrary. In general, these operators do not preserve the locality of elements (30) *of the symmetry algebra* $\operatorname{sym}\mathcal{E}_{\text{Toda}}$.

In Chapter 2, we construct the commutative Hamiltonian hierarchy $\mathfrak{A}$ of r-component analogs for the potential modified Korteweg–de Vries equation. This hierarchy is related with the Korteweg–de Vries hierarchy for equation (9) that describes the dynamics of the integral (27).

First, we consider an example ([48]) with $r = 1$. Namely, we consider the scalar Liouville equation

$$u_{xy} = \exp(2u)$$

and construct the symmetry sequence for this equation. This sequence is identified with the hierarchy $\mathfrak{A}$ of the potential modified Korteweg–de Vries equation

$$u_t = -\tfrac{1}{2}u_{xxx} + u_x^3$$

Also, we assign the hierarchy $\mathfrak{B}$ of the Korteweg–de Vries equation

$$T_t = -\tfrac{1}{2}T_{xxx} + 3TT_x$$

to the initial Liouville equation.

Now suppose $r \geq 1$. The hierarchy $\mathfrak{A}$ is constructed in Sec. 4 by using the following procedure. Introduce the nonlocal variable s such that $s_x = T$ and $s_y = 1$. Choose the initial function $\phi_{-1} = 1$. Then construct two sequences,

$$\phi_i = D_x^{-1}(\dot{T}_{\phi_{i-1}}),$$

see Eq. (31), and

$$\varphi_i = \square(\phi_{i-1}).$$

We denote the first sequence ϕ_i by $\mathfrak{B}$. This is the commutative bi–Hamiltonian hierarchy of local higher symmetries for the potential Korteweg–de Vries equation

$$s_t = -\beta\, s_{xxx} + \tfrac{3}{2}s_x^2, \qquad \beta = \sum_i a_i \Delta^i.$$

The second sequence φ_i is denoted by $\mathfrak{A}$. This is the required hierarchy of the higher analogs for the potential modified Korteweg–de Vries type equation

$$u_t = \varphi_1.$$

The following theorem is valid.

Theorem (theorem 41 on page 60)**.** *The elements φ_i of the sequence $\mathfrak{A} \subset \operatorname{sym} \mathcal{E}_{\text{Toda}}$ compose a commutative Lie algebra*:

$$[\mathfrak{A}, \mathfrak{A}] = 0.$$

Then we describe the elements $\varphi_k \in \mathfrak{A}$ such that $k < 0$. We prove that the symmetry φ_{-1} is defined by the Toda equation itself represented in the Hamiltonian form

$$u_y = A_1 \circ \mathbf{E}_u\bigl((\vec{a} \cdot \exp(Ku))\,\mathrm{d}x\bigr),$$

where

$$A_1 = \kappa^{-1} \cdot \bar{D}_x^{-1}$$

is the first Hamiltonian structure for the equation within the hierarchy $\mathfrak{A}$ and $\vec{a} = {}^t(a_1, \ldots, a_r)$, $\kappa = \|a_i k_{ij}\|$.

Theorem. 1 (theorem 47 on page 75). *The generators φ_k of the commutative Lie algebra $\mathfrak{A}$ are the Noether symmetries of the Toda equation*: $\varphi_k \in \operatorname{sym} \mathcal{L}_{\text{Toda}}$.

2 (proposition 49 on page 76). *Suppose $k \geq 0$ is arbitrary. Then the* kth *term $\phi_k = \mathbf{E}_T(h_k\,\mathrm{d}x)$ of the hierarchy $\mathfrak{B}$ is a conserved density for the* kth *higher potential modified Korteweg–de Vries equation.*

Now we formulate the most remarkable relation between the hierarchy $\mathfrak{A}$ for the potential modified Korteweg–de Vries equation (93) and the hierarchy $\mathfrak{B}$ for scalar equation (91).

Theorem. 1 (proposition 51 on page 79). *Each Noether symmetry*

$$\varphi_{\mathcal{L}} = \square \circ \mathbf{E}_T(Q(x, \mathbf{T})) \in \operatorname{sym} \mathcal{L}_{\text{Toda}}$$

of the Toda equation associated with a nondegenerate symmetrizable matrix K is Hamiltonian with respect to the Hamiltonian structure $A_1 = \kappa^{-1} \cdot D_x^{-1}$ and the Hamiltonian $\mathcal{H} = [Q(x, \mathbf{T})]$:

$$\varphi_{\mathcal{L}} = A_1 \circ \mathbf{E}_u(\mathcal{H}).$$

2 (theorem 52 on page 79). *The Hamiltonian $[h_k \, \mathrm{d}x]$ for the k*th *higher Korteweg–de Vries equation is the Hamiltonian for the Noether symmetry $\varphi_k \in \mathfrak{A}$ for any integer $k \geq 0$.*

In Chapter 3, we apply the methods of geometry of partial differential equations to the study of the properties of equations (5) and (3).

In Sec. 8 of Chapter 3 we obtain the geometric structures for the m-component analog

$$\Psi_t = \boldsymbol{i}\Psi_{xx} + \boldsymbol{i} f(|\Psi|)\, \Psi$$

of the nonlinear Schrödinger equation (3). By [98], this equation admits a commutative bi–Hamiltonian hierarchy of higher symmetries and an infinite set of the conserved densities in involution if $f = \mathrm{id}$. Suppose f is arbitrary, then this is not true. We compute the symmetry algebra for this equation if f is the homogeneous function of weight Δ; this case is actual for applications in physics. Also, we point out m^2 conserved currents

$$\eta_{ij} = \Psi^i \bar{\Psi}^j \, \mathrm{d}x + \boldsymbol{i} \left(\Psi^i_x \, \bar{\Psi}^j - \Psi^i \, \bar{\Psi}^j_x \right) \mathrm{d}t$$

for this equation. Here f is arbitrary. These currents generalize the known conservation laws of energy for an ith component Ψ^i in the multi–component nonliear Schrödinger equation.

In Sec. 9 of the same chapter we consider the dispersionless Toda equation (5),

$$u_{xy} = \exp(-u_{zz}),$$

and realize the following scheme for it. First, we find the Lie algebra of its classical symmetries and assign classes of exact solutions to these symmetries; then we construct conserved currents for this equation. The results of our calculations are rather cumbersome. They are found on pages 87–82 and also in the paper [50].

In Chapter 4, recent geometric concepts in the theory of Bäcklund transformations and zero–curvature representations are illustrated for the scalar Liouville equation

$$\mathcal{E}_{\text{Liou}} = \{u_{xy} = \exp(2u)\}, \tag{33}$$

which is the Toda equation associated with the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$.

In Sec. 10 we construct one–parametric families of Bäcklund (auto)-transformations $\tilde{\mathcal{E}}_t$ for equation (33). These transformations are

$$\tilde{\mathcal{E}}_t = \left\{ \begin{array}{l} (\tilde{u}-u)_x = \mathrm{e}^{-t}\,\exp(\tilde{u}+u), \\ (\tilde{u}+u)_y = 2\mathrm{e}^{t}\,\sinh(\tilde{u}-u) \end{array} \right\}. \tag{34}$$

By definition, put

$$u_k \equiv \frac{\partial^k u}{\partial x^k}, \quad u_{\overline{k}} \equiv \frac{\partial^k u}{\partial y^k}, \qquad k \in \mathbb{N}.$$

Consider the scaling symmetry

$$\hat{X} = -x\,\frac{\partial}{\partial x} + y\,\frac{\partial}{\partial y} + \sum_{k\geq 1} k u_k\,\frac{\partial}{\partial u_k} - \sum_{k\geq 1} k u_{\overline{k}}\,\frac{\partial}{\partial u_{\overline{k}}} \tag{35}$$

of the Liouville equation (33). Our reasonings are based on the following property of $\hat{X}$. This symmetry cannot be extended up to a symmetry of the covering equation $\tilde{\mathcal{E}}_t$. Recently I. S. Krasil'shchik described ([37]) the scheme of generating one–parametric families of coverings over differential equations. We have

Theorem ([37])**.** *Let $\tau\colon \tilde{\mathcal{E}} \to \mathcal{E}$ be a covering and $A_t\colon \tilde{\mathcal{E}} \to \tilde{\mathcal{E}}$ be a smooth family of diffeomorphisms such that $A_0 = \mathrm{id}$ and*

$$\tau_t = \tau \circ A_t\colon \tilde{\mathcal{E}} \to \mathcal{E}$$

is the covering for any $t \in \mathbb{R}$. Then the evolution of the Cartan connection form U_{τ_t} is

$$\frac{dU_{\tau_t}}{dt} = [\![\hat{X}_t, U_{\tau_t}]\!]^{\mathrm{FN}}, \tag{36}$$

where $\hat{X}_t$ is a τ_t-shadow for any $t \in \mathbb{R}$ and $[\![\cdot,\cdot]\!]^{\mathrm{FN}}$ is the Frölicher–Nijenhuis bracket, see Eq. (165) *on page 100.*

We prove that scaling symmetry (35) is the τ_t-shadow such that the evolution of the connection form U_t in the covering $\tau_t\colon \tilde{\mathcal{E}}_t \to \mathcal{E}_{\mathrm{Liou}}$, see Eq. (34), is given by Eq. (36). The proof is straighforward. It is based on a useful identity in total derivatives.

Theorem (lemma 62 on page 102)**.** *Let $u(x)$ and $f(u)$ be smooth functions and D_x be the total derivative with respect to x. Denote $u_k \equiv D_x^k(u(x))$, $k \geq 0$, $u_0 \equiv u$. Then the relation*

$$n \cdot D_x^n(f(u)) = \sum_{m=1}^{n} m\,u_m\,\frac{\partial}{\partial u_m} D_x^n(f(u))$$

holds for any integer $n \geq 1$.

In Sec. 11, we consider the problem of constructing the pairs of solutions to the hyperbolic Liouville equation (33) and the wave equation

$$s_{xy} = 0$$

such that these solutions are related by the Bäcklund transformation. Namely, we obtain the set of the nonlocal variables such that Bäcklund transformations are successfully integrated and each of the solutions related by Bäcklund transformation is obtained explicitly in terms of these variables.

In Sec. 12, we analyse the correspondence between Bäcklund transformations and zero–curvature representations. We use two distinct representations of the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ generated by the elements $\langle e,\, h,\, f\rangle$. The first representation is the representation of $\mathfrak{g}$ in the traceless matrices:

$$\varrho(e) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \varrho(h) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \varrho(f) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The second representation involves vector firlds on the straight line:

$$\rho(e) = 1 \cdot \frac{\partial}{\partial \Xi}, \qquad \rho(h) = -2\Xi \cdot \frac{\partial}{\partial \Xi}, \qquad \rho(f) = -\Xi^2 \cdot \frac{\partial}{\partial \Xi}.$$

We point out Bäcklund transformations assigned to the known zero–curvature representations, and *vice versa*.

In Appendix A, we list several geometrical methods of solving the boundary problems for equations of the mathematical physics. Constructing solutions invariant w.r.t. symmetries of the problem is discussed. A method based on the respesentation of the equation at hand in the evolutionary form is pointed out. The methods based on the deformation of the boundary problem along discrete or continuous parameters are described. Among these methods, we note the direct iterations, the boundary conditions homotopy, the relaxation method, and the deformation of the initial equation. Then, comparative analysis of the results of computer experiments in applying these methods is carried out.

Part I. The Korteweg–de Vries type equations associated with the Toda systems

In Part I, we analyse the properties of the symmetry algebra $\operatorname{sym}\mathcal{E}_{\text{Toda}}$ for the hyperbolic Toda equations $\mathcal{E}_{\text{Toda}}$ associated with nondegenerate symmetrizable matrices K. By using the canonical nonlocal recursion operator for the algebra $\operatorname{sym}\mathcal{E}_{\text{Toda}}$, we construct its commutative Lie subalgebra $\mathfrak{A}$ generated by the local higher Noether symmetries of the latter equation. We identify this subalgebra with the Hamiltonian hierarchy of the higher analogs for the r-component potential modified Korteweg–de Vries equation associated with K. Then we relate the hierarchy $\mathfrak{A}$ with the commutative bi-Hamiltonian hierarchy $\mathfrak{B}$ of the scalar Korteweg–de Vries equation. The hierarchy $\mathfrak{B}$ itself is a commutative Lie subalgebra of the Noether symmetries of the scalar wave equation. By using all these relations between the hyperbolic equations and the evolution hierarchies of their symmetries, we prove that the hierarchies $\mathfrak{A}$ and $\mathfrak{B}$ share the same sequence of the Hamiltonians and demonstrate that the Toda equation itself is the first nonlocal term in the symmetry algebra $\mathfrak{A}$.

Chapter 1. Conservation laws and the Noether symmetries of the Toda equations

In this chapter, we study the geometric properties of the two–dimensional Toda equations $u_{xy} = \exp(Ku)$ (in particular, of the Toda equations associated with the complex semisimple Lie algebras, see [66, 83]). Namely, we analyse the relations between their conservation laws ([68, 90]), Noether's symmetries of the Lagrangian functional ([86]), and the recursion operators for the symmetry algebra of the latter equations.

In Sec. 1, we pass from the scalar Liouville equation to the hyperbolic Toda equations associated with generic nondegenerate symmetrizable matrices K. Further, we consider properties of the Lagrangian for these equations. Then we point out the minimal integral $T \in \ker \bar{D}_y$ and its differential span $\mathbf{T} \subset \bar{D}_y$. Finally, we assign generators of the Lie algebra $\operatorname{sym}\mathcal{E}_{\text{Toda}}$ to elements $\boldsymbol{\Omega}$ of the kernel $\ker \bar{D}_y$.

In Sec. 2, we construct a one-to-one correspondence between the generating sections of conservation laws and the Noether symmetries of the Lagrangian $\mathcal{L}_{\text{Toda}}$ for the Toda equation. The description of the Lie algebra $\operatorname{sym}\mathcal{L}_{\text{Toda}} \subset \operatorname{sym}\mathcal{E}_{\text{Toda}}$ of the Noether symmetries for the Toda equation associated with a generic matrix K is based on this correspondence. Also, we establish some properties which are common to all integrals $\Omega^i \in \ker \bar{D}_y$ of the Toda equation associated with the Cartan matrix K of a semisimple Lie algebra.

Finally, in Sec. 3 we construct a continuum of the recursion operators for the symmetry algebra of the Toda equation. There are two types of

the recursion operators: they are either local or nonlocal with respect to the total dervatives.

The exposition follows the papers [54, 46, 57].

1. The Toda equation

The Liouville equation and its generalizations. The Liouville equation

$$\mathcal{E}_{\mathrm{Liou}} = \{u_{\xi\xi} + u_{\eta\eta} = \exp(2u)\} \tag{37}$$

is a model exactly integrable nonlinear differential equation that appeares in many parts of mathematics and mathematical physics. The study of this equation was initiated by Liouville ([69]) and H. Poincaré ([80]). One of the problems that they posed, the uniformization problem for algebraic curves (compact Riemannian surfaces), was investigated later by Kazdan and Warner ([41]). Suppose the Lagrangian

$$\mathcal{L}_{\mathrm{Liou}} = -\tfrac{1}{2}\big[\big(u_\xi^2 + u_\eta^2 + \exp(2u)\big)\,\mathrm{d}\xi \wedge \mathrm{d}\eta\big]$$

is regularized as described in ([101]) and then calculated on a classical solution of the Liouville equation. In this case, $\mathcal{L}_{\mathrm{Liou}}$ is known ([101]) to be the generating function for the accessor parameters that characterize the uniformization of a Riemannian surface of genus 0. Also, the Lagrangian $\mathcal{L}_{\mathrm{Liou}}$ for Eq. (37) is the potential for the Weil–Peterson metric on the Teichmüller space of marked Riemannian surfaces (*ibid*).

Equation (37) plays an important role in modern field theory. In particular, the quantum Liouville field appeares as a quantum anomaly in the string theory ([82]). Further, consider the free self–dual Yang–Mills equation

$$F_{\mu\nu} = *F_{\mu\nu},$$

where $F_{\mu\nu}$ is the stress tensor. Then, finding the N-instanton solutions, which minimize the action for the latter equation, also leads ([97]) to Eq. (37).

In Riemannian geometry, the Liouville equation (37) is the Gauss equation for a conformal metric on the Lobachevsky plane (see [23] and also [41]).

Example 10. Consider two pointwise equivalent Riemannian metrics

$$\mathrm{d}s_j^2 = f_j(x,y)(\mathrm{d}x^2 + \mathrm{d}y^2),$$

where $f_j > 0$ and j is either 1 or 2, on open twofolds of constant Gaussian curvature

$$K_j = -(2f_j)^{-1}\Delta \log f_j = \mathrm{const}_j\,.$$

Here Δ is the Laplace operator. Namely, suppose $f_2 = f_1 \cdot \exp(2u)$, then u satisfies the equation

$$\Delta u = -K_2 f_1 \exp(2u) - K_1 f_1. \tag{38}$$

Obviously, the two–dimensional elliptic Liouville equation corresponds to the case of the flat metric such that $f_1 \equiv 1$ (*i.e.*, $K_1 \equiv 0$) and the Lobachevsky plane metric ($K_2 \equiv -1$). This interpretation of Eq. (37) is treated as a limit case ($K_1 < 0$) in the superconductivity theory in description of the Abrikosov curls in the 2D model ([41]). The Liouville equation appeares in the study of the Kadomtsev–Pogutse equations. The latter equation is a reduction of the general magneto–hydrodynamic system, where some detailes that are inessential in the stability problem for high–temperature plasma in TOKAMAK are omitted ([33]).

For any solution of the elliptic Liouville equation (37) there is a Lobachevsky plane model that is conformally equivalent to the Euclidean plane with the diagonal metric $g_{ij} = \delta_{ij}$. Consider the standard Poincaré model in the upper half-plane $y > 0$ such that $f_2 = 1/y^2$. Then we get a particular solution $u = -\log y$ of Eq. (37).

Suppose $f_1 \equiv 1$, $K_1 \equiv 0$, and $K_2 \equiv +1$. Then formula (38) describes the conformal equivalence between the Eucludean metric δ_{ij} on the plane $\mathbb{E}^2$ and the metric $g_{ij} = \exp(2\mathcal{U})\,\delta_{ij}$ on the two-dimensional sphere S^2. The function $\mathcal{U}(x, y)$ satisfies the equation

$$\mathcal{U}_{xx} + \mathcal{U}_{yy} + \exp(2\mathcal{U}) = 0 \tag{39}$$

that differs from Eq. (37) by the sign at the exponent (or, equivalently, by the multiplication of the independent variables by $\boldsymbol{i} = \sqrt{-1}$). We shall say that Eq. (39) is the *Liouville* scal$^+$*-equation.*

Remark 4. Consider the orthogonal metric

$$\mathrm{d}s^2 = \alpha^2\,\mathrm{d}x^2 + \beta^2\,\mathrm{d}y^2$$

on the sphere of raduis ρ. Then, Eq. (39) is a particular (conformal) case of the equation ([18])

$$(\alpha^{-1}\beta_x)_x + (\beta^{-1}\alpha_y)_y + \rho^{-2}\alpha\beta = 0 \tag{40}$$

that describes an orthogonal coordinate net on this sphere. Indeed, $\alpha = \beta = \rho\exp(\mathcal{U})$ is a solution of Eq. (40) if $\mathcal{U}(x, y)$ is an arbitrary solution of Eq. (39). Obviously, this solution is not unique. The pair

$$\alpha = \sin\mathcal{V}, \quad \beta = \rho\mathcal{V}_y,$$

where $\mathcal{V}$ satisfies the sine-Gordon equation

$$\mathcal{V}_{xy} = \sin\mathcal{V},$$

or the pair

$$\alpha = \rho, \quad \beta = \rho\sin x$$

are also solutions to Eq. (40).

Suppose $\mathcal{E}$ is a differential equation. Throughout this article we denote the number of dependent variables u^j (the unknown functions) by r and suppose that these r functions depend on n base coordinates (usually they are denoted by $x^1, \ldots, x^n$).

In this paper, we consider the generalizations of the two–dimensional scalar Liouville equation (37) to the cases $n \geq 2$ and $r \geq 1$. We investigate the properties of these generalizations by using modern methods and tools of the PDE geometry.

We obtain a generalization of Eq. (37) for $n \geq 2$ by the following procedure. Recall that the Liouville equation is the condition for the Euclidean metric on $\mathbb{E}^n$ to be conformally equivalent to the conformal metric on an n-dimensional manifold of constant scalar curvature

$$\mathrm{scal} \equiv R = \mathrm{const}\,. \tag{41}$$

We fix the value $R = -2$ (such that the Gaussian curvature is $K = -1$ at $n = 2$) and preserve the correlation with the two–dimensional case of Eq. (37). We also put

$$\mathrm{d}s^2 = \exp(2u)\sum_k \mathrm{d}x^k \mathrm{d}x^k. \tag{42}$$

Relation (41) is a nonlinear PDE for the function $u(x^1, \ldots, x^n)$.

Theorem 8 ([46])**.** *Condition* (41) *is*

$$(n-1)\Delta u + \tfrac{1}{2}(n-1)(n-2)(\mathrm{grad}\, u)^2 = \exp(2u), \tag{43}$$

where Δ is the Laplace operator in the Euclidean space $\mathbb{E}^n$ and the scalar product $(\cdot,\cdot)$ is also induced by the Euclidean metric.

Proof. The scalar curvature R of metric (42) is given by the formula

$$R = \sum_{i,q} \exp(-2u) R^i_{qqi},$$

see [23]. Then we have

$$R^i_{qqi} = \partial_i\, \Gamma^i_{qq} - \partial_q\, \Gamma^i_{qi} + \Gamma^i_{pi}\Gamma^p_{qq} - \Gamma^i_{pq}\Gamma^p_{qi},$$

where

$$\Gamma^k_{ij} = \partial_i u\, \delta^k_j + \partial_j u\, \delta^k_i - \partial_l u\, \delta_{ij}\delta^{kl}$$

are the Christoffel symbols. After a straightforward calculation of the sums in q, i, and $p = 1 \ldots n$, we get (43). □

The Toda equations. M. Toda ([93]) considered the integrable nonlinear dynamical system described by Eq. (1), *i.e.*, the one–dimensional lattice with the exponential interaction. Nowadays, vastest literature is devoted to the study of the properties of system (1) and its various generalizations (see [22, 100, 66, 77] and references therein). In order to

obtain the two–dimensional Toda system out of the nonperiodic Toda lattice

$$q^i_{tt} = \frac{\partial H}{\partial q^i}, \qquad i \in \mathbb{Z},$$

where the Hamiltonian H is

$$H = -\sum_i \exp(q^i - q^{i+1}),$$

we replace the "acceleration" $\mathrm{d}^2/\mathrm{d}t^2$ with the operator $\partial^2/\partial x\partial y$. Different stages of this replacement process and the discovery of the Toda equations associated with the Lie algebras (and the Kac–Moody algebras as well, see [39]) are discussed in the papers [67, 65, 68, 66, 7, 8, 27, 28, 29], see also [22, 77].

In what follows, we derive the generalizations of Eq. (37) from a nontrivial viewpoint. The resulting number r of the dependent variables u^1, …, u^r is $r \geq 1$. Namely, we use the notion of the Laplace invariants ([100]). To start with, we simplify the calculations by making the change of coordinates such that the Liouville equation acquires the hyperbolic form $u_{xy} = \exp(u)$. From now on, we study the hyperbolic equations and their symmetries. The difference between the elliptic and the hyperbolic cases vanishes after the complexification. Still, the study of the symmetry properties of PDE does not require the presence of a complex structure. Therefore we assume all equations to be real in agreement with the definition on page 6.

Following [100], we calculate the principal Laplace invariants H_0 and H_1 for a hyperbolic equation

$$u_{xy} = f(x, y, u, u_x, u_y);$$

we have $f = \exp(u)$. Then we get

$$H_0 \stackrel{\text{def}}{=} -\bar{D}_y\left(\frac{\partial f}{\partial u_y}\right) + \frac{\partial f}{\partial u_x}\,\frac{\partial f}{\partial u_y} + \frac{\partial f}{\partial u} = \exp(u),$$

$$H_1 \stackrel{\text{def}}{=} -\bar{D}_x\left(\frac{\partial f}{\partial u_x}\right) + \frac{\partial f}{\partial u_x}\,\frac{\partial f}{\partial u_y} + \frac{\partial f}{\partial u} = \exp(u).$$

The definition of other Laplace's invariants follows from the equations

$$\bar{D}_{xy}(\log H_i) = -H_{i-1} + 2H_i - H_{i+1}, \quad i \in \mathbb{Z}. \tag{44}$$

We recall that the quasilinear equation $u_{xy} = f$ is of the *Liouvillean type* if its sequence of the Laplace invariants is finite, *i.e.*, there are some $p \geq 1$ and $q \geq 0$ such that $H_p = H_{-q} \equiv 0$. One easily checks that the sequence H_i for the Liouville equation vanishes at once: $H_i \equiv 0$ if i is neither 0 nor 1. This observation motivates the above definition[2]. Now suppose $-q < i < p$ and make the substitution $H_i = \exp(\mathcal{U}^i)$.

[2]We also note that the *matrix* Laplace invariants are defined for system (46) of hyperbolic differential equations, see [100].

Now, restrict equation (44) onto the graphs of jets of the sections $\mathcal{U}$ in the jet bundle $\mathbb{R}^r \times \mathbb{R}^2 \to \mathbb{R}^2$. Then the total derivatives $\bar{D}_i$ are mapped to the derivations $\partial/\partial x^i$. We finally obtain the system

$$\mathcal{U}^0_{xy} = 2\exp(\mathcal{U}^0) - \exp(\mathcal{U}^1), \quad \mathcal{U}^1_{xy} = -\exp(\mathcal{U}^0) + 2\exp(\mathcal{U}^1).$$

Generally, we obtain the system of equations

$$\mathcal{U}^i_{xy} = \sum_{j=-q+1}^{p-1} k_{ij}\,\exp(\mathcal{U}^j)$$

such that the structure of the nondegenerate $(p+q-1)\times(p+q-1)$-matrix $K = \|k_{ij}\|$ is

$$k_{ii} = 2, \qquad k_{i,i+1} = k_{i.i-1} = -1, \qquad k_{ij} = 0 \text{ if } |i-j| > 1. \tag{45}$$

We see that K is the Cartan matrix of the Lie algebra of type A_{r-1}. Now we shift the index i that enumerates the variables $\mathcal{U}^i$ and make the change ([66]) of the form $\mathcal{U} = K\cdot u$. Thus, we get the system

$$u^i_{xy} = \exp(-u^{i-1} + 2u^i - u^{i+1}), \quad 1 \le i \le r, \qquad u^0 = u^{r+1} \equiv 0.$$

This is the two–dimensional Toda system associated with the type A_{r-1} Lie algebra, which is denoted by $\mathfrak{g}$. Generally, we assign the Toda equations

$$\boldsymbol{u}_{xy} = \exp(K\boldsymbol{u}) \tag{46}$$

to a semisimple Lie algebra with the Cartan matrix K by following the geometric scheme ([66]), which is discussed below. In Chapter 4, we use the contructions supplied by this scheme and analyse the relationship between the canonical zero–curvature representations and Bäcklund transformations for a certain class of differential equations.

First, we fix some notation. Let $\mathfrak{g}$ be a semisimple complex Lie algebra of rank r. By $\{\alpha_i,\ 1 \le i \le r\}$ denote its system of simple roots and let K be the Cartan matrix of $\mathfrak{g}$. We thus have

$$K = \|k_{ij} = 2(\alpha_i, \alpha_j)\cdot|\alpha_j|^{-2},\ 1 \le i,\ j \le r\|.$$

By $K^{-1} = \|k^{ij}\|$ denote the inverse matrix for K and denote its elements by k^{ij}. Suppose A, $B \in \mathfrak{g}$. Then we assume that

$$\theta = A\,\mathrm{d}z + B\,\mathrm{d}\bar{z} \tag{47}$$

is the flat connection form in the principal fibre bundle $G \to M$, where G is the Lie group of the Lie algebra $\mathfrak{g}$. We thus have

$$\mathrm{d}\theta + \tfrac{1}{2}[\theta, \theta] = 0. \tag{48}$$

In terms of the extended total derivatives we have, then,

$$[\partial + A, \bar{\partial} + B] = 0 \quad \Leftrightarrow \quad -\bar{\partial}A + \partial B + [A, B] = 0. \tag{49}$$

Suppose further that H_j are the Cartan generators and E_j, F_j are the Chevalley generators of $\mathfrak{g}$, respectively, for all j such that $1 \leq j \leq r = \operatorname{rank}\mathfrak{g}$. The commutation relations

$$\begin{aligned} [H_i, H_j] &= 0, & [H_i, E_j] &= k_{ji}E_j, \\ [H_i, F_j] &= -k_{ji}F_j, & [E_i, F_j] &= \delta_{i,j}H_i \end{aligned} \tag{50}$$

hold. We choose the following ansatz for the connection coefficients A and B:

$$\begin{aligned} A &= \sum_{j=1}^{r} \left(a_h^j(z,\bar z)\cdot H_j + a_e^j(z,\bar z)\cdot E_j \right), \\ B &= \sum_{j=1}^{r} \left(b_h^j(z,\bar z)\cdot H_j + b_f^j(z,\bar z)\cdot F_j \right). \end{aligned} \tag{51}$$

Then from Eq. (49) we get the relations

$$\partial b_h^j - \bar\partial a_h^j + a_e^j b_f^j = 0, \tag{52a}$$

$$\bar\partial \log a_e^j = -\sum_{i=1}^{r} k_{ji}\, b_h^i, \qquad \partial \log b_f^j = \sum_{i=1}^{r} k_{ji}\, a_h^i \tag{52b}$$

for the coefficients of H_j, E_j, and F_j, respectively, for any j such that $1 \leq j \leq r$. From Eq. (52b) we obtain

$$a_h^i = \sum_{j=1}^{r} k^{ij}\,\partial \log b_f^j, \qquad b_h^i = -\sum_{j=1}^{r} k^{ij}\,\bar\partial \log a_e^j,$$

and from Eq. (52a) we get the relation

$$\sum_{j=1}^{r} k^{ij}\,\partial\bar\partial \log\left(a_e^j\cdot b_f^j\right) = a_e^i\cdot b_f^i \tag{53}$$

for any i, $1 \leq i \leq r$. By definition, put

$$u_i = \sum_{j=1}^{r} k^{ij}\,\log\left(a_e^j\cdot b_f^j\right). \tag{54}$$

Now substitute Eq. (54) in Eq. (53). We finally get the Toda equation (see Eq. (46)) associated with the Lie algebra $\mathfrak{g}$ ([66]). In the sequel, we use the coordinates x and y as synonims for the complex coordinates z and $\bar z$, respectively. Also, we suggest the following convention: all symmetries, conservation laws, and similar structures for the Toda equations are treated up to the discrete symmetry $x \leftrightarrow y$.

Now suppose $K = \|k_{ij},\ 1 \leq i,\ j \leq r\|$ is a nondegenerate $(r\times r)$-matrix and $K^{-1} = \|k^{ij}\|$ is its inverse. Assume that there is an r-component vector $\vec a$ such that $a_i \neq 0$ for any $1 \leq i \leq r$ and the symmetry condition $\kappa_{ij} \equiv a_i\cdot k_{ij} = \kappa_{ji}$ holds for the matrix $\kappa = \|\kappa_{ij}\|$. Then we say that this matrix K is *symmetrizable*, see also [89]. By

$\hat{\kappa}$ we denote the operator of left multiplication by the nondegenerate matrix κ.

The hyperbolic Toda equations associated with a nondegenerate symmetrizable $(r \times r)$-matrix K are

$$\mathcal{E}_{\text{Toda}} = \Bigl\{F^i \equiv u^i_{xy} - \exp\bigl(\sum_{j=1}^{r} k_{ij} u^j\bigr) = 0, \quad 1 \le i \le r\Bigr\}. \tag{55}$$

In particular, suppose $\mathfrak{g}$ is a semisimple Lie algebra of rank r, let $\{\alpha_i,$ $1 \le i \le r\}$ be the system of its simple roots, and denote the Cartan matrix of the algebra $\mathfrak{g}$ by

$$K = \left\| k_{ij} = \frac{2(\alpha_i, \alpha_j)}{|\alpha_j|^2},\ 1 \le i, j \le r \right\|.$$

Then we set $a_i = |\alpha_i|^{-2}$; we thus have

$$\kappa_{ij} = \frac{2(\alpha_i, \alpha_j)}{|\alpha_i|^2 \cdot |\alpha_j|^2} = \kappa_{ji}.$$

Lagrangian formalism for the Toda equation. The Toda equation $\mathcal{E}_{\text{Toda}}$ is the Euler-Lagrange equation in the following sence. Consider the action functional

$$\mathcal{L}_{\text{Toda}} = \int L_{\text{Toda}}\, \mathrm{d}x \wedge \mathrm{d}y$$

with the density

$$L_{\text{Toda}} = -\tfrac{1}{8} \sum_{i,j} \sum_{\mu,\nu} g^{\mu\nu} \kappa_{ij} u^i_{;\mu} u^j_{;\nu} + a_i^2 \cdot \exp\Bigl(\sum_{j=1}^{r} k_{ij} u^j\Bigr).$$

Here $u^j_{;\mu} \equiv D_\mu(u^i)$ and $g^{\mu\nu} = \left(\begin{smallmatrix} 0 & 2 \\ 2 & 0 \end{smallmatrix}\right)$ is the inverse of the metric tensor $g_{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$ that defines the flat metric $\mathrm{d}s^2 = \mathrm{d}x\,\mathrm{d}y$ on the base of the jet bundle π. In local coordinates, the density of the Lagrangian is

$$L_{\text{Toda}} = -\tfrac{1}{2} \sum_{i,j=1}^{r} \kappa_{ij} u^i_x u^j_y - \sum_{i=1}^{r} a_i \cdot \exp\Bigl(\sum_{j=1}^{r} k_{ij} u^j\Bigr). \tag{56}$$

Assign the Euler-Lagrange equations

$$\mathbf{E}_u(\mathcal{L}_{\text{Toda}}) = \Bigl|\sum_j \kappa_{ij} F^j\Bigr| = \kappa \cdot F = 0 \tag{57}$$

to the Lagrangian $\mathcal{L}_{\text{Toda}}$. Then these equations are equivalent to equation (55) since the matrices K and κ are nondegenerate simultaneously due to the assumption $a_i \neq 0$.

Suppose $\mathcal{E}$ is a differential equation, $\varphi \in \ker \bar{\ell}_{\mathcal{E}}$ is its symmetry and $\psi \in \ker \bar{\ell}^*_{\mathcal{E}}$ is the generating section of its conservation law. Now we obtain the transformation rules for φ and ψ with respect to the reparameterizations that preserve the equation's manifold $\mathcal{E}$ and the ideal $\mathcal{E}^\infty$ of its differential consequences. The following lemma is valid.

Lemma 9 ([57])**.** *Suppose $\mathcal{E} = \{G^i = 0,\ 1 \le i \le r\}$ be a nonoverdetermined differential equation and let*

$$G^i = A^i_j F^j$$

be a nondegenerate transformation of the relations that define the equation $\mathcal{E}$. Then the following two conditions hold:

(1) *The identities*

$$\ell_G = A \cdot \ell_F, \qquad \ell_G^* = \ell_F^* \cdot {}^t\!A,$$

are fulfilled. Here A is the reparameterization matrix for the relations that describe $\mathcal{E}$.

(2) *Suppose $\varphi_G \in \ker \bar{\ell}_G$ is a symmetry of the equation $\mathcal{E}$ and $\psi_G \in \ker \bar{\ell}_G^*$ is an arbitrary solution to Eq. (13) for $\mathcal{E} = \{G = 0\}$. Now, consider the reparameterization $G = AF$ of the relations that define the equation $\mathcal{E} \simeq \{F = 0\}$. Then $\varphi_F = \varphi_G$ is a symmetry of the equation $\mathcal{E}$ again: $\varphi_F \in \ker \bar{\ell}_F$, while a solution ψ_G of Eq. (13) is transformed by the rule*

$$\phi_G \mapsto \psi_F = {}^t\!A \cdot \psi_G \in \ker \bar{\ell}_F^*.$$

Proof. By using the definition of the linearization operator ℓ_G for $G = AF$, we get

$$\ell_G = \ell_{AF} = A \cdot \ell_F.$$

Therefore,

$$\ell_G^* = (A \cdot \ell_F)^* = \ell_F^* \circ A^* = \ell_F^* \circ {}^t\!A.$$

If the matrix A is nondegenerate, then the condition $\bar{\ell}_G(\varphi_G) = 0$ is equivalent to $A \cdot \bar{\ell}_F(\varphi_G) = 0$, whence we get $\varphi_F = \varphi_G$. From the assumption $\bar{\ell}_F^*({}^t\!A \cdot \psi_G) = 0$ we deduce the formula $\psi_F = {}^t\!A \cdot \psi_G$ for the rules of solution transformation for the equation $\bar{\ell}_F^*(\psi_F) = 0$. □

Now we apply the Noether theorem (see Theorem 6 on page 14):

Corollary 10. Let the assumptions of Theorem 6 and Lemma 9 hold. Let $\psi \in \ker \bar{\ell}_F^*$ be the generating section of a conservation law for the Euler-Lagrange equation $\mathcal{E} = \{F = 0\}$. Then there is the Noether symmetry $\varphi \in \ker \bar{\ell}_F$ of the equation $\mathcal{E}$ such that the relation $\psi = {}^t\!A \cdot \varphi$ holds.

The minimal integral for the Toda equation. Consider the Toda equation associated with a nondegenerate symmetrizable matrix K. One easily checks that Eq. (55) admits at least one *integral* ([100]), that is, an expession dependent of u^j_σ whose total derivative $\bar{D}_y$ vanishes on this equation. Namely, we have

$$T = \tfrac{1}{2}\sum_{i,j=1}^{r} \kappa_{ij} u^i_x u^j_x - \sum_{i=1}^{r} a_i \cdot u^i_{xx} \in \ker \bar{D}_y. \tag{58}$$

Both components T and $\bar{T}$ of the traceless energy–momentum tensor Θ for the Euler equations (57) are well known to be of form (58) up to the complex conjugation:

$$\Theta = T\,\mathrm{d}x + \bar{T}\,\mathrm{d}y,$$

see, *e.g.*, [77]. In Sec. 5 of Chapter 2 we shall discuss some aspects of the Hamiltonian formalism for the Toda equations and derive (58) from density (56) of the Lagrangian. Meanwhile, we construct the differential span $\mathbf{T}$ of the minimal integral $T \in \ker \bar{D}_y$. By definition, put

$$T_j \equiv \bar{D}_x^j(T).$$

The differential consequences T_j to the functional T generate the subspace $\mathbf{T} \subset \ker \bar{D}_y$ within the kernel of the total derivative $\bar{D}_y$. Indeed, any smooth function Q supplies the functional

$$Q(x, \mathbf{T}) \equiv Q(x, T, T_1, \ldots, T_\mu) \in \ker \bar{D}_y.$$

We say that the nondegenerate symmetrizable matrix K is *generic* if the integral T in Eq. (58) is a unique solution to the equation $\bar{D}_y(T) = 0$ on the corresponding Toda equation (55).

We emphasize that by a special choice of the matrix K one can achieve the situation such that the functional T will *not* be a unique integral for the Toda equation (55). The criteria for the equality $\dim\ker \bar{D}_y = 2$ to hold at $r = 2$ are found in the paper [68]. From now on, by $\mathbf{\Omega} \subseteq \ker \bar{D}_y$ we denote the subspace in $\ker \bar{D}_y$, which is differentially generated by *all* solutions Ω^i to the equation $\bar{D}_y(\Omega^i) = 0$. The total number of these solutions is denoted by q: $1 \leq i \leq q \leq r$. We always assume $\Omega^1 \equiv T$.

Example 11 ([68]). Consider the Cartan matrix $K = \left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right)$ of the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$. Then the Toda equation (55) associated with K admits two integrals. The first integral $\Omega^1 = T$, which is defined in Eq. (58), acquires the form

$$\Omega^1 = (u_x^1)^2 - u_x^1 u_x^2 + (u_x^2)^2 - u_{xx}^1 - u_{xx}^2, \tag{59a}$$

while the second integral is

$$\Omega^2 = u_{xxx}^1 + u_x^1 \cdot (u_{xx}^2 - 2u_{xx}^1) + (u_x^1)^2 \cdot u_x^2 - u_x^1 \cdot (u_x^2)^2. \tag{59b}$$

The integral Ω^2 is a solution to the equation $\bar{D}_y(\Omega) = 0$ but does not follow from Ω^1.

By [90], a necessary and sufficient condition for r nontrivial independent solutions Ω^i of the equation $\bar{D}_y(\Omega^i) = 0$ to exist is that K be the Cartan matrix of a semisimple Lie algebra $\mathfrak{g}$. The Toda equations associated with $\mathfrak{g}$ are exactly integrable ([65]).

The symmetry algebra of the Toda equations. In this subsection, we assign classes of infinitesimal symmetries of the Toda equation to the functional span $\boldsymbol{\Omega}$ of differential consequences to the integrals Ω^i, which were introduced in the previous subsection. Then, in Sec. 2 we find out which out these symmetries are the Noether symmetries of the Lagrangian $\mathcal{L}_{\text{Toda}}$.

By $\vec{\Delta} = |\Delta^i|$ denote the vector of the conformal dimensions ([82])

$$\Delta^i = \sum_{j=1}^{r} k^{ij}$$

of the Toda fields $\exp(u) \equiv {}^t(\exp(u^1), \ldots, \exp(u^r))$. This notation is well–defined due to

Proposition 11 ([8])**.** (1) *The transformation*

$$\begin{aligned} x &\mapsto \mathcal{X}(x), \\ y &\mapsto \mathcal{Y}(y), \\ u^i(x,y) &\mapsto \tilde{u}^i = u^i(\mathcal{X},\mathcal{Y}) + \Delta^i \,\log \mathcal{X}'(x)\mathcal{Y}'(y) \end{aligned} \tag{60}$$

is a finite conformal symmetry of the Toda equation $\mathcal{E}_{\text{Toda}}$.

(2) *The Lagrangian $\mathcal{L}_{\text{Toda}} = \int L_{\text{Toda}}\, \mathrm{d}x{\wedge}\mathrm{d}y$ is invariant with respect to this change.*

(3) *By definition, put $\beta \equiv \sum_{i=1}^{r} a_i \cdot \Delta^i$; under diffeomorphism* (60)*, the component T of the energy–momentum tensor Θ is transformed by the rule*

$$T[u] \mapsto \bigl(\mathcal{X}'(x)\bigr)^2 \cdot T[\tilde{u}(\mathcal{X},\mathcal{Y})] - \beta \cdot \Bigl(\frac{\mathcal{X}'''(x)}{\mathcal{X}'(x)} - \frac{3}{2}\Bigl(\frac{\mathcal{X}''(x)}{\mathcal{X}'(x)}\Bigr)^2\Bigr). \tag{61}$$

Remark 5. The symmetry properties, Eq. (60) and (61), of the Toda equation associated with the Lie algebras were considered in the paper [8]. The present formulation of Proposition 11 is an extension of the cited result to the Toda equation (55), which is now associated with an arbitrary nondegenerate symmetrizable $r \times r$-matrix K. The coefficients $\vec{\Delta}$ and β are already adapted to this general case.

The infinitesimal variant of Proposition 11 is

Proposition 12. 1 ([75, 77]). *The infinitesimal components of conformal symmetries* (60) *of the Toda equation are*

$$\varphi_0^f = \square(f(x))$$

up to the transformation $x \leftrightarrow y$. Here f is an arbitrary smooth function and

$$\square = \vec{u}_x + \vec{\Delta} \cdot \bar{D}_x \tag{62}$$

is a vector–valued, first–order differential operator.
2 ([45]). *Each point symmetry* φ_0^f *is a Noether symmetry of the Lagrangian* $\mathcal{L}_{\text{Toda}}$:

$$\mathbf{Э}_{\varphi_0^f}(L_{\text{Toda}}\,\mathrm{d}x) \in \operatorname{im} \mathrm{d}_h.$$

3 ([77, 7]). *The functional* T*, which was defined in Eq.* (58)*, is a density of the Hamiltonian for the infinitesimal conformal symmetry* φ_0^f:

$$\varphi_0^f = A_1 \cdot \mathbf{E}_u\bigl(T \cdot f(x)\,\mathrm{d}x\bigr), \qquad A_1 = \hat{\kappa}^{-1} \cdot D_x^{-1}. \tag{63}$$

The third statement in Proposition 12 was formulated in local coordinates in the paper [77]. In Sec. 5 of Chapter 2 we establish the relation between the canonical Hamiltonian formalism for the Toda equation and the Hamiltonian operator A_1 above. In Theorem 52 on page 52 we find out that equality (63) is the root part of the infinite sequence of relations between the hierarchy of the higher analogs of the Korteweg–de Vries equation, see Eq. (9) on page 11, and the commutative hierarchy within the Noether symmetries algebra for the Toda equation (this hierarchy is constructed in Sec. 4).

Lemma 13. *Suppose* K *is a symmetrizable* $r \times r$*-matrix and the operator* $\square$ *is defined by Eq.* (62)*. Then the relations*

$$\begin{aligned} \bar{\ell}_F \circ \square &= \bar{D}_x \circ \square \circ \bar{D}_y, \\ \bar{\ell}_F^* \circ \hat{\kappa} \circ \square &= \bar{D}_x \circ \hat{\kappa} \circ \square \circ \bar{D}_y \end{aligned} \tag{64}$$

hold.

Proof. Here we express the first relation in local coordinates:

$$\begin{aligned} \bar{\ell}_F \circ \square &= \Bigl\| \delta_{ij}\,\bar{D}_{xy} - k_{ij} \exp\Bigl(\sum_l k_{il}u^l\Bigr) \Bigr\| \cdot \bigl| u_x^j + \Delta^j \cdot \bar{D}_x \bigr| = \\ &= \boldsymbol{u}_x\,\bar{D}_{xy} + \boldsymbol{u}_{xx}\,\bar{D}_y + \boldsymbol{u}_{xy}\,\bar{D}_x + \boldsymbol{u}_{xxy} + \vec{\Delta}\,\bar{D}_{xxy} - \\ &- \Bigl| \sum_j k_{ij}u_x^i \exp\Bigl(\sum_l k_{il}u^l\Bigr) \Bigr| - \Bigl| \sum_{j,p} k_{ij}k^{jp} \exp\Bigl(\sum_l k_{il}u^l\Bigr) \Bigr| \cdot \bar{D}_x = \\ &= \bar{D}_x \circ \bigl| \boldsymbol{u}_x + \vec{\Delta}\,\bar{D}_x \bigr| \circ \bar{D}_y, \end{aligned}$$

Q. E. D.

The second relation is deduced from the first one by using Lemma 9 and the Helmholtz condition

$$\bar{\ell}_{\mathbf{E}(\mathcal{L}_{\text{Toda}})} = \bar{\ell}^*_{\mathbf{E}(\mathcal{L}_{\text{Toda}})}$$

that holds since the matrix $\kappa = {}^t\kappa$ is symmetric. $\square$

Corollary 14. The vector–functions

$$\varphi = \square(\phi(x, \boldsymbol{\Omega})) \tag{65}$$

are symmetries of the Toda equation: $\varphi \in \operatorname{sym} \mathcal{E}_{\text{Toda}}$ for any function ϕ that depends on an arbitrary subset $\boldsymbol{\Omega}$ of the integrals $\Omega_j^i \equiv \bar{D}_x^j(\Omega^i) \in \ker \bar{D}_y$.

Formula (65) gives the description of the whole symmetry algebra $\operatorname{sym}\mathcal{E}_{\text{Toda}}$:

Proposition 15 ([75]). (1) *Let the assumptions of Lemma* 13 *hold; then any symmetry of Eq.* (55) *is* (65).

(2) *Suppose further K is such that there exist q independent integrals $\Omega^i \in \ker \bar{D}_y$, where $1 < q \leq r$. Put*

$$f^i = \exp(\sum_{j=1}^{r} k_{ij}u^j) \text{ and } f^i_j = k_{ij} \cdot f^i.$$

Assume further that there are r constant $(r \times q)$-matrices $M_\alpha = \|(M_\alpha)^i_j\|$, $1 \leq i \leq r$, $1 \leq j \leq q$, and a constant $(r \times q)$-matrix $\Delta = \|\Delta^i_j\|$, $1 \leq i \leq r$, $1 \leq j \leq q$, such that $\operatorname{rank}\Delta = q$ *and the equations*

$$\Delta^\beta_i\, f^\alpha_\beta = (M_\beta)^\alpha_i\, f^\beta, \qquad f^\alpha_\beta\,(M_\gamma)^\beta_i = (M_\beta)^\alpha_i\, f^\beta_\gamma$$

hold. Now, construct the first–order, $(r \times q)$-matrix differential operator

$$\square = \sum_{\alpha=1}^{r} M_\alpha \cdot u^\alpha_x + \Delta \cdot \bar{D}_x \tag{66}$$

and consider a vector $\phi = |\phi^i(x, \mathbf{\Omega})|$, $1 \leq i \leq q$. Under these assumptions, the sections defined in Eq. (65) *exhaust all symmetries of the Toda equation $\mathcal{E}_{\text{Toda}}$.*

Therefore, in both cases we have

$$\operatorname{sym}\mathcal{E}_{\text{Toda}} \simeq \{\varphi^i = \square^i_j \phi^j(x, \mathbf{\Omega}) \bmod (x \leftrightarrow y)\},$$

where the number of columns in the operator $\square$ is equal to the number q of the independent integrals Ω^l and ϕ is an arbitrary vector.

Corollary 16. Any solution ψ to the equation $\bar{\ell}^*_F(\psi) = 0$ for system (55) is of the form

$$\psi = \hat{\kappa}\big(\square(\phi(x, \mathbf{\Omega}))\big). \tag{67}$$

We stress that the problem of finding the integrals $\mathbf{\Omega}$ is primary for the Toda equations, and the search for the symmetries φ as well as the selection of the Noether symmetries $\varphi_{\mathcal{L}}$ follow this problem. We also note that each conformal symmetry (60) of the Toda equation is Noether, *i.e.*, preserves the Lagrangian $\mathcal{L}_{\text{Toda}}$. Still, not each section φ of type (65) is a Noether symmetry of Eq. (55).

2. The Noether symmetries of the Toda equation

First, we recall Example 3 on page 12 and obtain an important property of the Toda equation. This property permits the application of the generating sections machinery in description of the conservation laws and the Noether symmetries of the Toda equation. Namely, we have

Lemma 17. *The Toda equation $\mathcal{E}_{\mathrm{Toda}}$ is ℓ-normal.*

Proof. By Example 3 on page 12, it is sufficient to represent the equation $\mathcal{E}_{\mathrm{Toda}}$ in the evolutionary form. Let

$$\xi = x + y, \quad \eta = x - y$$

be the new independent variables. The way to choose them is such that the equations

$$u^i_{\xi\xi} - u^i_{\eta\eta} = \exp\Bigl(\sum_{j=1}^{r} k_{ij} u^j\Bigr)$$

hold. Now, put $v^i \equiv u^i_\eta$; then the equations $\mathcal{E}_{\mathrm{ev}} \subset J^2(\mathbb{R}^2, \mathbb{R}^{2r})$ of the form

$$\Bigl\{u^i_\eta = v^i, \qquad v^i_\eta = u^i_{\xi\xi} - \exp\Bigl(\sum_{j=1}^{r} k_{ij} u^j\Bigr)\Bigr\}$$

are the required evolutionary representation of Eq. (55). $\square$

From relation (57) and Corollary 10 we deduce the relation $\psi = \kappa\,\varphi_{\mathcal{L}}$ between the Noether symmetries and the generating sections of conservation laws for the Toda equations. This observation allows to specify a property common for all the integrals Ω^i for Eq. (55). Namely, suppose

$$\mathrm{d}_h(\Omega^i\,\mathrm{d}x) = -\tilde{\nabla}_i(F)\,\mathrm{d}x \wedge \mathrm{d}y$$

for any admissible i and consider the conservation law $[\eta] = \bigl[Q(x, \mathbf{\Omega})\,\mathrm{d}x\bigr]$. Then we have

$$\mathrm{d}_h Q(x, \mathbf{\Omega})\,\mathrm{d}x = -\sum_{i,j} \frac{\partial Q}{\partial \Omega^i_j}\, D_x^j \circ \tilde{\nabla}_i(F)\,\mathrm{d}x \wedge \mathrm{d}y.$$

By definition, the generating section ψ_η of the conservation law $[\eta]$ is

$$\psi_\eta = -\sum_{i,j} (-1)^j \bigl(\tilde{\nabla}_i\bigr)^* \circ D_x^j \left(\frac{\partial Q}{\partial \Omega^i_j}\right) = -\sum_i \bigl(\tilde{\nabla}_i\bigr)^* \circ \mathbf{E}_{\Omega^i}(Q). \tag{68}$$

Now we compare Eq. (67) with Eq. (68) and, by using Theorem 6 and Lemma 9, we obtain

Theorem 18. (1) *Suppose Ω^i is an integral for the Toda equation* (55) *such that $D_y(\Omega^i) = \tilde{\nabla}_i(F)$. Then there is the operator ∇_i such that*

$$\tilde{\nabla}_i = \nabla_i \circ \square^* \circ \hat{\kappa}.$$

In particular, consider the integral $\Omega^1 = T$ defined in Eq. (58)*. Then*

$$\nabla_1 = \mathbf{1}.$$

(2) *The Noether symmetries of the Toda equation are*

$$\varphi_{\mathcal{L}} = \square \circ \sum_i \nabla_i^* \circ \mathbf{E}_{\Omega^i}(Q(x, \boldsymbol{\Omega})),$$

where $\Omega^i \in \ker \bar{D}_y$ *are the integrals for the equation* $\mathcal{E}_{\text{Toda}}$,

$$\mathbf{E}_{\Omega^i} = \sum_{j\geq 0} (-1)^j D_x^j \cdot \frac{\partial}{\partial \Omega^i_j}$$

is the Euler operator with respect to Ω^i, $\boldsymbol{\Omega}$ *is an arbitrary set of differential consequences to* Ω^i, *and* Q *is a smooth function.*

Example 12. Again, consider the Toda equation (55) associated with the root system A_2. Then we have $r = 2$, $a_i = |\alpha_i|^{-2} = 1$ for $i = 1, 2$, and

$$K = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad K^{-1} = \frac{1}{3} \cdot \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad \vec{\Delta} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Put $\square = \vec{u}_x + \vec{\Delta} \cdot \bar{D}_x$, see (62). The integrals Ω^1 and Ω^2 are defined in Eq. (59). One easily checks that $D_y(\Omega^1) = \square^* \circ \hat{\kappa}(F)$. Therefore, $\nabla_1 = \mathbf{1}$ and

$$D_y(\Omega^2) = -D_x \circ \square^* \circ \hat{\kappa}(F),$$

thence, $\nabla_2 = -D_x$. We emphasize that the integral Ω^2 is *not* equivalent to $-D_x(\Omega^1)$.

Remark 6. In Theorem 18 we established the relation $\nabla_1 = \mathbf{1}$ to hold for the minimal integral T, which is defined in Eq. (58). Now we restrict all previous reasonings onto the subspace $\{Q(x,\ \mathbf{T})\} \subset \ker \bar{D}_y$ of the kernel of the total derivative. Thus, we consider the subspace that is generated by the integral T and its differential consequences. Then, by Theorem 6, the conservation laws $[Q\,\mathrm{d}x]$ for the Toda equation $\mathcal{E}_{\text{Toda}}$ are in one-to-one correspondence with the Noether symmetries

$$\varphi_{\mathcal{L}} = \square \circ \mathbf{E}_T(Q(x, \mathbf{T})). \tag{69}$$

In other words, the Noether symmetries $\varphi_{\mathcal{L}}$ of the Toda equation (55) associated with a generic nondegenerate symmetrizable matrix K are (69) up to the transformation $x \leftrightarrow y$. This statement is an extension of the relation ([86]) between the Noether symmetries and conservation laws for the scalar Liouville equation onto $r \geq 1$. We note that our scheme of reasonings is essentially more simple than the straightforward proof for the Liouville equation (*ibid*).

In the rest part of this section, we generalize the method of the paper [86]. We solve the equation

$$\mathfrak{I}_\varphi(\mathcal{L}_{\text{Toda}}) = 0$$

directly with respect to φ and thus find out which of the symmetries $\varphi = \square(\phi(x,\ \mathbf{T})) \in \operatorname{sym} \mathcal{E}_{\text{Toda}}$ preserve the Lagrangian $\mathcal{L}_{\text{Toda}}$ for the Toda equations. We follow the paper [54].

Detailed analysis of some algebraic aspects for Eq. (55) shows (technically, we calculate the term $E_2^{0,2}(\mathcal{E}_{\text{Toda}})$ of the Vinogradov's $\mathcal{C}$-spectral sequence, see [94]) that the following two conditions are equivalent:

$$\ni_\varphi(\mathcal{L}_{\text{Toda}}) = 0 \quad \Longleftrightarrow \quad \mathbf{E}_u(\ni_\varphi(\mathcal{L}_{\text{Toda}})) = 0,$$

i.e., a density is a total direvgence iff its variation is zero. We note that this question was not discussed in [86]. Further on, we get

Lemma 19. *The relation* $\mathbf{E}_u(\ni_\varphi(\mathcal{L}_{\text{Toda}})) = 0$ *is equivalent to the condition*

$$-\mathbf{E}_u\left(D_y(T)\cdot\phi(x,T,\ldots,T_m)\right) = 0,$$

where $\varphi = \Box(\phi)$ *and the total derivative* D_y *is* not *restricted onto* $\mathcal{E}_{\text{Toda}}$.

The proof of Lemma 19 is based on multiple use of the relation

$$\sum_{j=1}^{r} \kappa_{ij}\Delta^j = a_i.$$

Lemma 20. *The highest order* m *of the derivative* T_m *in the set of* ϕ*'s arguments is even*: $m = 2\mu$, *and* $\phi = \phi(x,T,\ldots,T_{2\mu})$.

Proof. Put $G_i = \dfrac{\delta}{\delta u^i}(\ni_\varphi(\mathcal{L}_{\text{Toda}}))$ and by $u^j_{(k,l)}$ denote the derivative $D_x^k \circ D_y^l(u^j)$. Then we have

$$\frac{\partial G_i}{\partial u^j_{(m+4,1)}} = -\left[(-1)^3 + (-1)^{m+2}\right]\cdot a_i\, a_j\cdot \frac{\partial\phi}{\partial T_m} = 0,$$

whence $(-1)^3 + (-1)^{m+2} = 0$ and thus m is even. $\Box$

Lemma 21. *The derivatives*

$$\frac{\partial^2\phi}{\partial T_m \partial T_l}$$

vanish for all l *such that* $\mu < l \leq m$.

Proof. It suffices to check that

$$\frac{\partial G_i}{\partial u^j_{2m+4}},\quad \frac{\partial G_i}{\partial u^j_{2m+2}},\quad \ldots,\frac{\partial G_i}{\partial u^j_{m+6}}$$

are equal to 0. This can be done straighforwardly. $\Box$

Still, the derivative

$$\frac{\partial^2\phi}{\partial T_\mu \partial T_{2\mu}}$$

is, in general, nontrivial owing to

Lemma 22. *The identity*

$$\mathbf{E}_u(-D_y(T)\cdot\mathbf{E}_T(P(x,T,\ldots,T_\mu))) \equiv 0$$

holds for any function P.

Proof. Indeed, we have

$$D_y(T)\cdot\mathbf{E}_T(P)\,\mathrm{d}x\wedge\mathrm{d}y = \langle D_y(T), \ell_P^{(T)^*}(1)\rangle = \langle \ell_P^{(T)}(D_y(T)), 1\rangle + \mathrm{d}_h\gamma = \langle \mathrm{Э}_{D_y(T)}^{(T)} P(x,T,\ldots,T_\mu), 1\rangle + \mathrm{d}_h\gamma = \langle D_y(P), 1\rangle + \mathrm{d}_h\gamma \in \ker \mathbf{E}_u,$$

where the coupling $\langle\,,\,\rangle$ takes values in the horizontal 2-forms $\omega = f\cdot \mathrm{d}x\wedge\mathrm{d}y$, $\mathrm{d}_h = \sum_i \mathrm{d}x^i\otimes D_i$ is the horizontal differential, $\mathrm{d}_h\gamma\in\ker\mathbf{E}_u$ is an exact form, and both the evolutionary derivation and the linearizations are evaluated with respect to T. □

Proposition 23. *A symmetry $\varphi = \square(\phi(x,\,T,\,\ldots,\,T_m)) \in \ker \bar{\ell}_F$ is a Noether symmetry of Toda's equation* (55) *iff the following two conditions hold*:

1. $m = 2\mu$, $\mu \geq 0$, *and*
2. *the function $\phi = \mathbf{E}_T(Q(x,T,\ldots,T_\mu)) \in \operatorname{im}\mathbf{E}_T$ lies in the image of the Euler operator with respect to T, where Q is an arbitrary smooth function.*

Proof. We show that $\phi\in\operatorname{im}\mathbf{E}_T$ by induction. Choose $P = P(m;x,T,\ldots,T_\mu)$ such that

$$\frac{\partial^2 P(m)}{\partial T_\mu^2} = (-1)^\mu\cdot\frac{\partial\phi}{\partial T_m} \tag{70}$$

and put $\tilde\phi \overset{\text{def}}{=} \phi - \mathbf{E}_T(P(m))$, then we get

$$\frac{\partial\tilde\phi}{\partial T_m} \equiv 0.$$

By Lemma 20, the equation

$$\mathbf{E}_u(D_y(T)\cdot\tilde\phi(x,T,\ldots,T_{m-2})) = 0$$

holds. By using Lemma 21, we choose $P(m)$ in accordance with (70) and apply Lemma 22. Therefore, we decrease the order $m = 2\mu$ to 0 with the step 2 inductively. Finally, we obtain

$$\phi = \mathbf{E}_T\Bigl(\sum_{i=0}^{\mu} P(2i;x,T,\ldots,T_i)\Bigr).$$

The proof is complete. □

Thus, we have obtained another description of the Noether symmetries' class (69) for the Toda equation assigned to a generic matrix K.

3. Recursion operators for the Toda equation

In this section, we construct a continuum of the recustion operators, which are either local or nonlocal with respect to D_x, for the symmetry algebra of the Toda equation. Although the structure of the symmetry algebra itself is known, see Eq. (65), presence of the recursion operators gives us additional information about the Toda equation and permits

to establish the relation between $\mathcal{E}_{\mathrm{Toda}}$ and other mathematical physics equations.

The explicit method by J. Krasil'shchik that allows construction of the recursion operators for symmetry algebras of differential equations was briefly described in the Introduction on pages 15–18. The theorem below contains the result of application of this method to the Toda equation (55) associated with a nondegenerate symmetrizable $(r \times r)$-matrix.

Theorem 24 ([57]). (1) *Equation* (55) *admits a continuum of* local *recursion operators* $R\colon \operatorname{sym}\mathcal{E}_{\mathrm{Toda}} \to \operatorname{sym}\mathcal{E}_{\mathrm{Toda}}$*, which are*

$$R = \square \circ \sum_{i,j} f_{ij}(x, \mathbf{\Omega}) \cdot \bar{D}_x^j \circ \ell_{\Omega^i}.$$

Here f_{ij} are arbitrary smooth functions and the linearizations ℓ_{Ω^i} with respect to the integrals Ω^i for the Toda equation are

$$\ell_{\Omega^i} = \Bigl(\ldots, \underbrace{\sum_{\sigma} \frac{\partial \Omega^i}{\partial u^k_\sigma} \cdot \bar{D}_\sigma}_{k\text{th } component}, \ldots\Bigr). \tag{71}$$

(2) *There is a continuum of* nonlocal *recursion operators for Eq.* (55)*, which are constructed in the following way. Assign the nonlocal variables s^i to the integrals Ω^i by the compatible differentiation rules $s^i_x = \Omega^i$ and $s^i_y = 0$. The linearizations ℓ_{s^i} are defined by the formulas*

$$\ell_{s^i} = \bar{D}_x^{-1} \circ \ell_{\Omega^i},$$

where ℓ_{Ω^i} are calculated by Eq. (71)*. Then the required recursion operators are*

$$R = \square \circ \sum_i f_i(x, \mathbf{s}, \mathbf{\Omega}) \cdot \bar{D}_x^{-1} \circ \ell_{\Omega^i},$$

where the functions f_i are arbitrary. In general, these operators do not preserve the locality of elements (65) *of the symmetry algebra* $\operatorname{sym}\mathcal{E}_{\mathrm{Toda}}$.

Proof. First, enlarge the set of the dependent variables u^j_σ by introducing the nonlocals s^i such that their derivatives are

$$s^i_x = \Omega^i, \qquad s_y = 0. \tag{72}$$

(In fact, any definition of s^i_y is allowed if it is compatible with the condition $s^i_{xy} = s^i_{yx} = 0$). Then, extend the total derivatives:

$$\tilde{D}_x = \bar{D}_x + \sum_i \Omega^i \frac{\partial}{\partial s^i}, \qquad \tilde{D}_y = \bar{D}_y,$$

such that $[\tilde{D}_x, \tilde{D}_y] = 0$. The Cartan's flat connection is now defined on the equation

$$\tilde{\mathcal{E}}^\infty = \{\tilde{D}_x^{k+1}(s^i) = \bar{D}_x^k(\Omega^i),\ \tilde{D}_x^k \circ \tilde{D}_y(s^i) = 0,\ k \geq 0; \\ \bar{D}_\sigma(F) = 0,\ |\sigma| \geq 0\}.$$

The Cartan's generating 1-forms

$$\omega_{\text{Toda}} \in C^\infty(\tilde{\mathcal{E}}^\infty) \otimes \mathcal{C}\Lambda^1(\tilde{\mathcal{E}}^\infty)$$

for the recursion operators R_{Toda} satisfy the determining equation

$$\tilde{\ell}^{[1]}_{\text{Toda}}(\omega_{\text{Toda}}) = 0,$$

where $\tilde{\ell}^{[1]}_{\text{Toda}}$ is the restriction of the linearization of Eq. (55) onto $H^{1,0}_{\mathcal{C}}(\tilde{\mathcal{E}})$. From the factorization

$$\bar{\ell}_{\text{Toda}} \circ \square = \bar{D}_x \circ \square \circ \bar{D}_y$$

in Eq. (64) it follows that any Cartan's 1-form

$$\omega_{\text{Toda}} = \square\Big(\sum_{i\geq 0} f_i(x, \mathbf{s}, \ldots, \tilde{D}_x^{k_i}(s^j)) \cdot \mathrm{d}_{\mathcal{C}}\big(\tilde{D}_x^i(s^l)\big)\Big) \tag{73}$$

is an element of the kernel $\ker \tilde{\ell}^{[1]}_{\text{Toda}}$, whence we get the statement of the theorem. In particular, suppose that each f_i does not depend on the nonlocal variables $\mathbf{s}$. Then the resulting recursion operator is local. $\square$

Example 13. Consider the integral T defined in Eq. (58). Its linearization is

$$\ell_T = \Big(\ldots, \underbrace{\sum_{j=1}^{r} \kappa_{ij} u_x^j \cdot \bar{D}_x - a_i \cdot \bar{D}_x^2}_{i\text{th component}}, \ldots\Big) = \square^* \circ D_x \circ \hat{\kappa}. \tag{74}$$

Now, introduce the nonlocal variable s: we set $s_x = T$ and $s_y = 1$. The compatibility condition for the variable s is

$$s_{xy} = 0. \tag{75}$$

Finally, we construct the recursion operator

$$R_{\text{Toda}} = \square \circ \bar{D}_x^{-1} \circ \ell_T.$$

Apply the operator R_{Toda} to the translation $u_x \in \operatorname{sym} \mathcal{E}_{\text{Toda}}$. We obtain the symmetry sequence

$$\varphi_k = \square(\phi_{k-1})$$

that corresponds to the sequence of functions

$$\phi_{-1} = 1, \quad \phi_0 = s_1, \quad \phi_1 = -\beta s_3 + \tfrac{3}{2} s_1^2, \\ \phi_2 = \beta^2 s_5 - \tfrac{5}{2}\beta s_2^2 - 5\beta s_1 s_3 + \tfrac{5}{2} s_1^3,$$

etc. In Chapter 2 we shall investigate the properties of this recursion operator R_{Toda} together with the properties of the symmetry sequence

$$\mathfrak{A} = \{\varphi_k \equiv R^k_{\text{Toda}}(\varphi_0),\ \varphi_0 = \vec{u}_x\}$$

in detailes. Meanwhile, we claim that the symmetries φ_k of the Toda equation, which are obtained by multiple action of R_{Toda} to the translation $\varphi_0 = \vec{u}_x$, are local, Hamiltonian, and commute with each other. Also, in the next section we shall establish the relation between the symmetry sequence $\mathfrak{A}$, the Korteweg–de Vries equations (9) and (16), and recursion operators (18) for the latter equations.

Chapter 2. The Korteweg–de Vries hierarchies and the Toda equations

In this chapter, we construct the commutative Hamiltonian hierarchy $\mathfrak{A}$ of multi–component analogs for the potential modified Korteweg–de Vries equation. This hierarchy is identified with the commutative Lie subalgebra in $\operatorname{sym}\mathcal{E}_{\text{Toda}}$ composed by certain Noether's symmetries. Also, we discuss some aspects of the Hamiltonian formalism for the Toda equations themselves and establish a link between the hierarchy $\mathfrak{A}$ and the higher Korteweg–de Vries equations for Eq. (9). The exposition follows the papers [51, 60].

We start our study of the relation between the Toda equation (55) and classical equations (9) and (16) of mathematical physics, the Korteweg–de Vries equations, with the following

Example 14. Consider the hyperbolic Liouville equation

$$\mathcal{E}_{\text{Liou}} = \{u_{xy} - \exp(2u) = 0\}. \tag{76}$$

The minimal integral, see Eq. (58), for the latter equation is ([99, 100])

$$T = u_1^2 - u_2, \quad \bar{D}_y(T) = 0. \tag{77}$$

Introduce the nonlocal variable s such that

$$s_x = T, \qquad s_y = 1, \tag{78}$$

and put

$$\vartheta \equiv 2u_1. \tag{79}$$

Then, consider the symmetry

$$\varphi = (u_1 + \tfrac{1}{2}\bar{D}_x)(T)$$

of the Liouville equation and calculate the evolution of the variables u, ϑ, T, and s along this symmetry. We get

$$u_t = -\tfrac{1}{2}u_3 + u_1^3 \qquad \text{(potential mKdV)} \tag{80}$$

$$T_t = -\tfrac{1}{2}T_3 + 3TT_1 \qquad \text{(KdV)} \tag{81}$$

$$s_t = -\tfrac{1}{2}s_3 + \tfrac{3}{2}s_1^2 \qquad \text{(potential KdV).} \tag{82}$$

The Miura transformation (see [76, 84]) acquires the form

$$\vartheta_1 = \mp 2T \mp \tfrac{1}{2}\vartheta^2, \qquad \vartheta_t = \pm T_2 - (\vartheta T_1 + \vartheta_1 T) \tag{83a}$$

$$T = \mp\tfrac{1}{2}\vartheta_1 - \tfrac{1}{4}\vartheta^2. \tag{83b}$$

One can treat relations (83) as Bäcklund transformation between the Korteweg–de Vries equation, see Eq. (81), and the equation

$$\vartheta_t = -\tfrac{1}{2}\vartheta_3 + \tfrac{3}{4}\vartheta^2\vartheta_1 \qquad \text{(modified KdV).} \tag{84}$$

The signs '$\pm$' and '$\mp$' in Eq. (83) are induced by the symmetry $\vartheta \mapsto -\vartheta$ of equation (84). This discrete transformation provides Bäcklund autotransformation ([10, 95]) for Eq. (81).

The recursion operator

$$R_{\text{Liou}} = D_x^2 - 2u_1 + D_x^{-1} u_1\, D_x$$

is common for both equations (76) and (80), see [48, 40]. This operator generates the commutative Lie subalgebra

$$\mathfrak{A}_{\text{Liou}} = \operatorname{span}_{\mathbb{R}}\langle \varphi_k = R_{\text{Liou}}^k(\varphi_0),\ \varphi_0 = u_1\rangle$$

of local higher symmetries of the potential modified Korteweg–de Vries equation, see Eq. (80).

Now, to the Liouville equation we assign the variable $\mathfrak{v}$ such that

$$\mathfrak{v}_x = \exp(2u), \tag{85a}$$

$$\mathcal{E}_{\mathfrak{v}} = \{\mathfrak{v}_y = \mathfrak{v}^2\} \tag{85b}$$

and therefore the compatibility condition $\mathfrak{v}_{xy} = \mathfrak{v}_{yx}$ holds. Thence, the equation $\mathcal{E}_{\text{Liou}}$ is represented in the evolutionary form:

$$u_{t_{-1}} = \varphi_{-1} \equiv \mathfrak{v}, \tag{86}$$

where the variable $t_{-1} \equiv y$ is the parameter and φ_{-1} is the shadow of the nonlocal symmetry

$$\tilde{\ni}_{\varphi_{-1},a_{-1}} \equiv \tilde{\ni}_{\varphi_{-1}} + a_{-1} \cdot \frac{\partial}{\partial \mathfrak{v}}$$

such that $a_{-1} = \mathfrak{v}^2$. The solution to Eq. (85b) is $\mathfrak{v} = -(y + \mathcal{X}(x))^{-1}$. Apply transformation (60) of the form $\tilde{y} = \mathcal{Y}(y)$ to the "time" y. Then we obtain the potential

$$\mathfrak{v} = -\frac{\mathcal{Y}'(y)}{(\mathcal{X}(x) + \mathcal{Y}(y))}$$

for the genreral solution $u = \frac{1}{2}\log \mathfrak{v}_x$ of the Liouville equation (see [69, 85]):

$$u = \tfrac{1}{2}\log\Bigl[\frac{\mathcal{X}'(x)\cdot\mathcal{Y}'(y)}{(\mathcal{X}(x) + \mathcal{Y}(x))^2}\Bigr]. \tag{87}$$

We recall that functional (77) is continuous on formal divergences $u \to \pm\infty$ of solution (87), see the paper [46] and references therein. We estimate the scheme of constructing the general solution for the Liouville equation by using the potential $\mathfrak{v}$ to be really laconic and productive. A similar approach was used in [10] to derive the Cole–Hopf transformation for the Burgers equation, see Eq. (203) on page 122.

Solution (87) of Eq. (76) is the mapping

$$\tau\colon \bigl\{\mathcal{X}_y = 0,\ \mathcal{Y}_x = 0\bigr\} \to \mathcal{E}_{\text{Liou}}.$$

Consider the evolutionary vector field $\ni_{u_t}$ defined in Eq. (80). This field can be lifted onto the inverse image of the covering τ. Hence we obtain the equation $\mathcal{Y}_t = 0$ and the Krichever–Novikov equation

$$\mathcal{X}_t = -2\mathcal{X}_3 + 3\mathcal{X}_2^2/\mathcal{X}_1 = -2\mathcal{X}_1 \cdot \{\mathcal{X}, x\}, \tag{88}$$

where $\{\mathcal{X}, x\}$ is the Schwartz derivative. Also, the evolution equation

$$\mathfrak{v}_t = a_1 \equiv \mathfrak{v}_3 - \tfrac{3}{2}\mathfrak{v}_2^2\mathfrak{v}_1^{-1}$$

holds.

The evolution equations that appear in Example 14 are systemized as shown in the diagram below:

$$\begin{tikzcd}
\boxed{\begin{array}{c}\text{The Krichever-}\\ \text{Novikov Eq. (20)}\end{array}} \arrow[d, "\text{Eq.}"', "\text{(19)}"] & & \boxed{\begin{array}{l}\text{The Krichever-}\\ \text{Novikov Eq. (88),}\\ \mathcal{Y}_t = 0\end{array}} \arrow[d, "\text{Eq.}"', "\text{(87)}"] \\
\boxed{\begin{array}{c}\text{Potential}\\ \text{KdV Eq. (82)}\end{array}} \arrow[rr, "\text{Eq.}", "\text{(72)}"'] \arrow[d, "\text{Eq.}"', "\text{(78)}"] & & \boxed{\begin{array}{c}\text{Petential}\\ \text{mKdV Eq. (80)}\end{array}} \arrow[d, "\text{Eq.}"', "\text{(79)}"] \\
\boxed{\text{KdV Eq. (81)}} & & \boxed{\begin{array}{c}\text{Modified}\\ \text{KdV Eq. (84)}\end{array}} \arrow[ll, "\text{Eq.}"', "\text{(83)}"]
\end{tikzcd}.$$

4. Analogs of the potential modified Korteweg–de Vries equation

4.1. **Constructing the hierarchy $\mathfrak{A}$.** First, we formulate an important property of the integrals $\Omega^i \in \ker D_y\big|_{\mathcal{E}}$ for a Liouvillean type equation $\mathcal{E}$. Namely, we specify the evolution of these integrals along symmetries $\varphi \in \operatorname{sym}\mathcal{E}$ of the equation $\mathcal{E}$.

Lemma 25 ([100])**.** *Consider a symmetry field Э_φ of a Liouvillean type equation $\mathcal{E}$. Then the evolution $\text{Э}_\varphi(\Omega^i)$ of an arbitrary integral $\Omega^i \in \ker \bar{D}_y$ for $\mathcal{E}$ is an element of* $\ker \bar{D}_y$ *again.*

Proof. Indeed, we have

$$\bar{D}_y(\text{Э}_\varphi(\Omega^i)) = \text{Э}_\varphi(\bar{D}_y(\Omega^i)) = \text{Э}_\varphi(0) = 0.$$

□

Example 15. Consider a symmetry $\varphi = \square(\phi(x, \mathbf{T}))$ of the Toda equation, see Eq. (65) on page 39. Then the relation

$$\dot{T}_\phi \equiv \text{Э}_{\square(\phi)}(T) = \bigl(-\beta\,\bar{D}_x^3 + T\,\bar{D}_x + D_x \circ T\bigr)(\phi) \tag{89}$$

holds. Here

$$\beta \equiv \sum_{i=1}^{r} a_i \cdot \Delta^i, \quad \square = \boldsymbol{u}_x + \vec{\Delta}\,\bar{D}_x, \text{ and } \Delta^i = \sum_j k^{ij}.$$

Also, consider the nonlocal variable s that was defined in Example 13 by the equations $s_x = T$ and $s_y = 1$. Then the evolution s_t of the

nonlocality is described by the formula

$$\ni_{\vec{\varphi}}(s) = D_x^{-1} \circ \ni_{\vec{\varphi}}(T).$$

Suppose that the sequence $\vec{\varphi}_0$, $\vec{\varphi}_1$, $\vec{\varphi}_2$ of symmetries of the Toda equation is assigned to the specially chosen set of the functions ϕ in formula (65). We set $\phi_{-1} = 1$ and obtain the symmetry $\varphi_0 = \Box(\phi_{-1})$. Then, we calculate the evolution $\dot{T}_{\phi_{-1}}$ that corresponds to this symmetry. Now we equal the next function ϕ_0 to the evolution of the potential s. Then, we obtain the function ϕ_1 and the symmetries φ_1, φ_2 in a similar way. The result is shown in the next diagram:

$$\begin{tikzcd}
\vec{\varphi}_2 & \phi_1 = -\beta s_3 + \frac{3}{2}s_1^2 \arrow[l, "\Box"'] & \dot{T}_{\phi_0} = -\beta T_3 + 3TT_1 \arrow[l, "D_x^{-1}"'] & \\
 & \vec{\varphi}_1 = \Box(s_1) \arrow[ul, "R_{\mathrm{Toda}}"] \arrow[ur, "\ell_T"] & \phi_0 = s_1 \arrow[l, "\Box"'] & \dot{T}_{\phi_{-1}} = T_1 \arrow[l, "D_x^{-1}"'] \arrow[ul, "R_{\mathrm{KdV}}"'] \\
 & & \vec{\varphi}_0 = \boldsymbol{u}_x \arrow[ul, "R_{\mathrm{Toda}}"] \arrow[ur, "\ell_T"] & \phi_{-1} = 1 \arrow[l, "\Box"']
\end{tikzcd} \tag{90}$$

The following evolution equations are met in diagram (90): the Korteweg–de Vries equation is

$$\mathcal{E}_{\mathrm{KdV}} = \{T_t = -\beta T_3 + 3T \cdot T_1\} \tag{91}$$

(we recall that its recursion operator is (18a) on page 16); then, we also get the potential Korteweg–de Vries equation

$$\mathcal{E}_{\mathrm{pKdV}} = \left\{s_t = -\beta s_3 + \tfrac{3}{2}s_1^2\right\}, \tag{92}$$

and the equation

$$\mathcal{E}_{\mathrm{pmKdV}} = \{\boldsymbol{u}_t = \Box(T(\boldsymbol{u}_1, \boldsymbol{u}_2))\}. \tag{93}$$

Suppose $K = \|2\|$ is the Cartan matrix of the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$. Then the Toda equation (55) associated with this algebra is the hyperbolic Liouville equation (76), and equation (93) is nothing else than the scalar potential modified Korteweg–de Vries equation (82). Further on, suppose that the matrix K in Eq. (55) is an arbitrary nondegenerate symmetrizable $(r \times r)$-matrix, not necessarily the Cartan matrix of a semisimple Lie algebra $\mathfrak{g}$ of rank r. Then we get the r-component system of the third-order evolution equations with the cubic nonlinearity. In local coordinates, this system is

$$u_t^i = \tfrac{1}{2}\sum\nolimits_{p,q=1}^{r} a_p \cdot \left\{k_{pq}u_x^i u_x^p u_x^q + 2(\Delta^i k_{pq} - \delta_{i,q})u_{xx}^p u_x^q - 2\Delta^i u_{xxx}^p\right\},$$

where $1 \le i \le r$; we recall that

$$\Delta^i = \sum_j k^{ij} \text{ and } a_p k_{pq} = a_q k_{qp}.$$

The analogs of the potential modified Korteweg–de Vries equation for $r = 2$*.* In this subsection, we consider evolution systems (93) in case $r = 2$ and the matrix K is symmetric.

Example 16. Let K be the symmetric matrix

$$K = \begin{pmatrix} 2 & \lambda \\ \lambda & 2 \end{pmatrix},$$

thence,

$$\vec{\Delta} = \frac{1}{\lambda+2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Introduce new dependent variables

$$u = u^1 + u^2, \quad v = (\lambda+2)(u^1 - u^2);$$

the inverse transformation is defined by the formulas

$$u^1 = \frac{1}{2}u + \frac{1}{2(\lambda+2)}v, \qquad u^2 = \frac{1}{2}u - \frac{1}{2(\lambda+2)}v.$$

Then, the normal form of Eq. (93) is

$$\begin{aligned} u_t &= -\frac{2}{\lambda+2}u_3 + \frac{2-\lambda}{(\lambda+2)^3}v_1v_2 + \frac{\lambda+2}{4}u_1^3 + \frac{2-\lambda}{4(\lambda+2)^2}u_1v_1^2, \\ v_t &= -u_2v_1 + \frac{\lambda+2}{4}u_1^2v_1 + \frac{2-\lambda}{4(\lambda+2)^2}v_1^3. \end{aligned} \tag{94}$$

Further on, make the scaling transformation

$$t = (\lambda+2)^2 \cdot \tilde{t}, \qquad u = (\lambda+2)^{-1} \cdot \tilde{u}, \qquad v = \tilde{v}, \tag{95}$$

preserving the same notation x, t, u, and v.

Proposition 26. *In coordinates* (95)*, equations* (94) *become linear in* λ *and acquire the form*

$$\begin{aligned} u_t &= -2u_3 + 2v_1v_2 + \tfrac{1}{2}u_1^3 + \tfrac{1}{2}u_1v_1^2 + \lambda\Big(-v_1v_2 + \tfrac{1}{4}u_1^3 - \tfrac{1}{4}u_1v_1^2\Big), \\ v_t &= -2u_2v_1 + \tfrac{1}{2}u_1^2v_1 + \tfrac{1}{2}v_1^3 + \lambda\Big(-u_2v_1 + \tfrac{1}{4}u_1^2v_1 - \tfrac{1}{4}v_1^3\Big). \end{aligned} \tag{96}$$

Remark 7. Consider the vector coefficient of λ in Eq. (96). Then this vector-valued function is *not* a symmetry of the flow that stands at λ^0 (and *vice versa*). Namely, the following statements hold.

Proposition 27. (1) *Consider the flow*

$$\begin{aligned} u_t &= -2u_3 + 2v_1v_2 + \tfrac{1}{2}u_1^3 + \tfrac{1}{2}u_1v_1^2, \\ v_t &= -2u_2v_1 + \tfrac{1}{2}u_1^2v_1 + \tfrac{1}{2}v_1^3 \end{aligned}$$

at λ^0 *in Eq.* (96)*. Then its symmetries*

$$\begin{pmatrix} \varphi^1, \\ \varphi^2 \end{pmatrix}(t, x, u, v, u_1, v_1, u_2, v_2, u_3, v_3)$$

of order ≤ 3 are generated by

$$\begin{pmatrix} tu_t + \frac{1}{3}xu_x \\ tv_t + \frac{1}{3}xv_x \end{pmatrix}, \qquad \begin{pmatrix} u_t \\ v_t \end{pmatrix}, \qquad \begin{pmatrix} u_x \\ v_x \end{pmatrix}, \qquad \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

These generators are the scaling symmetry, two translations, and two shifts, respectively. The commutator with the scaling symmetry maps the translations along t and x to themselves.

(2) *Consider the flow*

$$u_t = -v_1 v_2 + \tfrac{1}{4}u_1^3 - \tfrac{1}{4}u_1 v_1^2,$$
$$v_t = -u_2 v_1 + \tfrac{1}{4}u_1^2 v_1 - \tfrac{1}{4}v_1^3$$

at λ^1 in Eq. (96)*. Then the symmetries*

$$\begin{pmatrix} \varphi^1 \\ \varphi^2 \end{pmatrix} (t, x, u, v, u_1, v_1, u_2, v_2, u_3, v_3)$$

of order ≤ 3 of this flow are generated by the sections

$$\begin{pmatrix} tu_t + \frac{1}{3}xu_x \\ tv_t + \frac{1}{3}xv_x \end{pmatrix}, \qquad \begin{pmatrix} u_t \\ v_t \end{pmatrix}, \qquad \begin{pmatrix} u_x \\ v_x \end{pmatrix}, \qquad \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

which are the scaling, the translations, and the shifts, respectively.

The conservation laws for system (96) in the normal form are few.

Proposition 28. *Suppose λ is generic* (*see Remark* 8 *below*)*. Then there is a unique generating section*

$$\begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} (t, x, u, v, u_1, v_1, u_2, v_2) = \begin{pmatrix} u_2 \\ \frac{\lambda-2}{\lambda+2}\, v_2 \end{pmatrix}$$

of order ≤ 2 of a conservation law for the flow in Eq. (96); *this section corresponds to the conserved density*

$$H = u_x^2 + \frac{\lambda - 2}{\lambda + 2}\, v_x^2.$$

In Sec. 7 we shall demonstrate that equation (94) is Hamiltonian with respect to the operator $A_1 = K^{-1} \cdot \bar{D}_x^{-1}$ (recall that the matrix K is assumed symmetric and therefore $a_i = 1$). Now we get the Hamiltonian representation of system (96) by using the transformation rule for Hamiltonian operators with respect to transformations of the dependent variables.

Lemma 29. *Consider a Hamiltonian equation*

$$u_t = A\big(\mathbf{E}_u(\mathcal{H}[u])\big).$$

Let $\tilde{u} = Qu$ be a nondegenerate transformation of the dependent variables. Then the equation

$$\tilde{u} = \tilde{A}\big(\mathbf{E}_{\tilde{u}}(\mathcal{H}[\tilde{u}])\big)$$

holds for the Hamiltonian operator

$$\tilde{A} = Q \cdot A \cdot {}^tQ.$$

Example 17. Consider Eq. (96). Then we get the diagonal operator

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = 2 \begin{pmatrix} \lambda + 2 & 0 \\ 0 & \frac{(\lambda+2)^2}{2-\lambda} \end{pmatrix} \cdot D_x^{-1}\bigl(\mathbf{E}_{(u,v)}(T^2\,\mathrm{d}x)\bigr),$$

where the density h_1 of the Hamiltonian $T^2\,\mathrm{d}x$ for the Korteweg–de Vries equation acquires the following form in the coordinates u, v:

$$h_1 = \tfrac{1}{16}\bigl((\lambda-2)v_x^2 - (\lambda+2)u_x^2 + 4(\lambda+2)u_{xx}\bigr)^2.$$

Proposition 30. *Again, suppose λ is generic* (*see Remark* 8). *The symmetries*

$$\begin{pmatrix} \varphi^1 \\ \varphi^2 \end{pmatrix}(t, x, u, v, u_1, v_1, u_2, v_2)$$

of order ≤ 2 of equation (96) *are generated by the translation and the shifts, which are*

$$\begin{pmatrix} u_x \\ v_x \end{pmatrix}, \qquad \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{97}$$

respectively.

Remark 8. We emphasize that the symmetry algebra (and even the classical symmetry algebra) of system (96) depends on the initial matrix K essentially. The cases $\lambda = -1$ (the matrix K corresponds to the Lie algebra A_2) and $\lambda = \pm 2$ (the matrix K is degenerate) are special. In what follows, we analyse them separately.

First, let $\lambda = -1$, that is, K is the Cartan matrix for the Lie algebra A_2. Then substitution (95) is

$$u = u^1 + u^2, \qquad v = u^1 - u^2.$$

After routine transformations, we obtain

$$\begin{aligned} u_t &= -u_3 + \tfrac{3}{2}v_1v_2 + \tfrac{1}{8}u_1^3 + \tfrac{3}{8}u_1v_1^2, \\ v_t &= \qquad\;\; - \tfrac{1}{2}u_2v_1 + \tfrac{1}{8}u_1^2v_1 + \tfrac{3}{8}v_1^3. \end{aligned} \tag{98}$$

Proposition 31. (1) *The symmetries*

$$\begin{pmatrix} \varphi^1 \\ \varphi^2 \end{pmatrix}(t, x, u, v, u_1, v_1, u_2, v_2)$$

of order ≤ 2 of flow (98) *are generated by the sections*

$$\begin{pmatrix} v \\ -\frac{1}{3}u + \frac{2}{3}\log v_x \end{pmatrix}, \qquad \begin{pmatrix} u_x \\ v_x \end{pmatrix}, \qquad \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The first symmetry is nonpolynomial. *The other two are the translation and the shift, respectively. The nonpolynomial symmetry flow can be also represented in the form*

$$u_{xt}^2 = \exp(u + 3u_{tt}).$$

(2) *The generating sections*

$$\begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} (t, x, u, v, u_1, v_1, u_2, v_2)$$

of order ≤ 2 *of conservation laws for Eq.* (98) *are*

$$\begin{pmatrix} u_2 \\ 3v_2 \end{pmatrix}, \qquad \begin{pmatrix} \Psi(u, v, v_x) \\ 2\partial\Psi/\partial v + 2v_2 v_1^{-1}\,\partial\Psi/\partial v_1 - u_1\,\partial\Psi/\partial v_1 \end{pmatrix},$$

where the function Ψ *satisfies the equation*

$$\frac{\partial\Psi}{\partial u} + \frac{1}{2}v_1\,\frac{\partial\Psi}{\partial v_1} - \frac{1}{2}\Psi = 0.$$

Now, for the first time within this paper, suppose that the matrix K is degenerate. Quite untrivially, this degeneracy can occur in two distinct ways.

First, assume that $\lambda = 2$ and $K = \left(\begin{smallmatrix} 2 & 2 \\ 2 & 2 \end{smallmatrix}\right)$. Then system (94) is in triangle form[3]:

$$\begin{cases} u_t = -\frac{1}{2}u_3 + u_1^3, \\ v_t = -u_2 v_1 + u_1^2 v_1. \end{cases} \tag{99}$$

It consists of the potential modified Korteweg–de Vries equation and the auxiliary dispersionless component. We note that the variable v admits an arbitrary shift. Indeed, the section ${}^t(0,\ f(v))$ is a symmetry of Eq. (99) if f is arbitrary. In addition, there are two more symmetries. They are the translation and the shift of u, see Eq. (97).

Proposition 32. *If* $\lambda = 2$, *then the generating sections*

$$\begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} (t, x, u, v, u_1, v_1, u_2, v_2)$$

of order ≤ 2 *of conservation laws for the flow in Eq.* (99) *are*

$$\begin{pmatrix} u_2 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} \exp(u) \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} \exp(-u) \\ 0 \end{pmatrix}.$$

The corresponding conserved densities are

$$\rho_1 = -\tfrac{1}{2}u_x^2, \quad \rho_2 = \exp(u), \quad \rho_3 = -\exp(-u),$$

[3]The classical symmetry algebra for the Toda equations associated with this matrix is composed by the sections

$$\vec{\varphi} = \alpha\vec{u}_x + \beta(x)\,\vec{\mathbf{1}},$$

where the constant α and the function $\beta(x)$ are arbitrary.

respectively. The last two densities belong to the nonpolynomial *conserved currents*

$$\eta_2 = \exp(u)\,\mathrm{d}x + (2u_2 - u_1^2)\exp(u)\,\mathrm{d}t,$$
$$\eta_3 = -\exp(-u)\,\mathrm{d}x + (2u_2 + u_1^2)\exp(-u)\,\mathrm{d}t$$

for the potential modified Korteweg–de Vries equation

$$u_t = -2u_3 + u_1^3$$

whose right–hand side is polynomial.

Now, suppose $\lambda = -2$, whence $K = \left(\begin{smallmatrix} 2 & -2 \\ -2 & 2 \end{smallmatrix}\right)$. Then equation (96) is

$$\begin{cases} u_t = -2u_3 + 4v_1v_2 + u_1v_1^2, \\ v_t = v_x^3. \end{cases} \tag{100}$$

Proposition 33. *The symmetries of system* (96) *for* $\lambda = -2$ *are described by Eq.* (97)*. The generating sections*

$$\begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} (t, x, u, v, u_1, v_1, u_2, v_2)$$

of order ≤ 2 *of conservation laws for the flow in right-hand side of Eq.* (100) *are, then,*

$$\begin{pmatrix} 0 \\ \Psi(x, t, v, v_x, v_{xx}) \end{pmatrix}.$$

Here the function Ψ *is subject to the equation*

$$\frac{\partial \Psi}{\partial t} - 2v_1^3\,\frac{\partial \Psi}{\partial v} - 3v_1^2\,\frac{\partial \Psi}{\partial x} + v_1v_2^2\,\frac{\partial \Psi}{\partial v_2} - 6v_1v_2\,\Psi = 0.$$

Factorizations of the recursion operators. Now we return to the problem of constructing the commutative hierarchy $\mathfrak{A}$, which is associated with the initial Toda equation $\mathcal{E}_{\mathrm{Toda}}$.

By construction, the symmetries φ_i are related by the recursion operator

$$R_{\mathrm{Toda}} = \square \circ D_x^{-1} \circ \ell_T \tag{101}$$

for the Toda equation (55). We met this operator in Example 13 on page 46. We fix[4] this notation R_{Toda} onwards.

In this subsection, we demonstrate that each symmetry

$$\vec{\varphi}_k = \square(\phi_{k-1}) \in \operatorname{sym} \mathcal{E}_{\mathrm{Toda}}$$

defines the next function ϕ_k explicitly by the relation $\phi_k = \ni_{\vec{\varphi}_k}(s)$, and therefore, $\vec{\varphi}_{k+1} = \square(\phi_k)$. We also prove that the sections ϕ_k compose the hierarchy of the potential Korteweg–de Vries equation (92).

[4]As we have already noted in Chapter 1, we treat the Toda equations, as well as the structures on them, up to the symmetry $x \leftrightarrow y$.

First, we note that the functions ϕ_{-1}, ϕ_0, and ϕ_1 are mapped successfully one to another by the recursion operator ([96])

$$R_{\mathrm{pKdV}} = -\beta\, D_x^2 + 2s_1 - D_x^{-1} \circ s_2 \tag{102}$$

for equation (92). Recall that this operator was obtained in Introduction while illustrating the Cartan generating forms method ([62]) by I. S. Krasil'shchik.

Lemma 34. *The following two decompositions of the recursion operators hold*:

$$R_{\mathrm{Toda}} = \square \circ \ell_s, \qquad R_{\mathrm{pKdV}} = \ell_s \circ \square, \tag{103}$$

where $\ell_s = D_x^{-1} \circ \ell_T$ *by Eq.* (23) *and the linearization* ℓ_T *is defined in Eq.* (74).

Proof. The first decomposition holds by construction. To establish another one, we use the relations

$$\begin{aligned}
&\ell_T \circ (\boldsymbol{u}_x + \vec{\Delta} D_x) = \\
&= \tfrac{1}{2}\sum_{i,j} \kappa_{ij} u_1^i\, D_x \circ u_1^j + \tfrac{1}{2}\sum_i a_i \Big\{ \Big(\sum_j k_{ij} u_1^i\Big) D_x \circ u_1^i - 2D_x^2 \circ u_1^i \Big\} + \\
&\qquad + \tfrac{1}{2}\sum_{i,j} \kappa_{ij} u_1^i \Delta^j D_x^2 + \tfrac{1}{2}\sum_i a_i \Big\{ \Big(\sum_j k_{ij} u_1^j\Big) \Delta^i D_x^2 - 2\Delta^i D_x^3 \Big\} = \\
&= -\sum_i a_i \Delta^i\, D_x^3 + \tfrac{1}{2}\sum_{i,j} a_i \big\{ k_{ij}(u_1^i u_2^j + u_2^i u_1^j) - 2u_3^i \big\} + \\
&\quad + \tfrac{1}{2}\sum_{i,j} a_i \big\{ k_{ij}(u_1^i u_1^j + u_1^i u_1^j) - 4u_3^i \big\}\, D_x + \\
&\quad + \tfrac{1}{2}\Big\{ \sum_i a_i \cdot (-2u_1^i) D_x^2 + \sum_{i,j} \kappa_{ij} \Delta^j u_1^i D_x^2 + \sum_{i,j} \kappa_{ij} \Delta^i u_1^j D_x^2 \Big\} = \\
&\qquad\qquad\qquad\qquad\qquad\qquad = -\beta D_x^3 + T_1 + 2T\, D_x,
\end{aligned}$$

whence follows the second decomposition. $\square$

We draw attention to the fact that the representation of the scalar operator R_{pKdV} in the form of the product of the *vector–valued* operator $\square$ and the row ℓ_s of length r seems to have been unnoticed in the literature.

Now we identify infinitesimal symmetries of differential equations with autonomous evolution equations and, by using Lemma 34, we extend diagram (90) infinitely upwards. The result is displayed in the

diagram

$$
\begin{tikzcd}
\cdots & \cdots \\
\boldsymbol{u}_{t_2} = \vec{\varphi}_2 \arrow[u, "R_{\mathrm{Toda}}"] \arrow[ur, "\ell_s"] & s_{t_1} = \phi_1 = -\beta s_3 + \frac{3}{2}s_1^2 \arrow[l, "\Box"'] \arrow[u, "R_{\mathrm{pKdV}}"'] \\
\boldsymbol{u}_{t_1} = \vec{\varphi}_1 \arrow[u, "R_{\mathrm{Toda}}"] \arrow[ur, "\ell_s"] & s_{t_0} = \phi_0 = s_1 \arrow[l, "\Box"'] \arrow[u, "{R_{\mathrm{pKdV}}= -\beta D_x^2+2s_1-D_x^{-1}\circ s_2}"'] \\
\underbrace{\boldsymbol{u}_{t_0} = \vec{\varphi}_0 = \boldsymbol{u}_x}_{\mathfrak{A}} \arrow[u, "R_{\mathrm{Toda}}"] \arrow[ur, "\ell_s"] & \underbrace{s_{t_{-1}} = \phi_{-1} = 1.}_{\mathfrak{B}} \arrow[l, "\Box"'] \arrow[u, "R_{\mathrm{pKdV}}"']
\end{tikzcd}
\tag{104}
$$

The right-hand sides ϕ_k of the evolution equations $s_{t_k} = \phi_k$ are generated by the recursion operator R_{pKdV}, therefore they are higher symmetries of the potential Korteweg–de Vries equation $\mathcal{E}_{\mathrm{pKdV}}$.

The times t_i in the equations $u_{t_k} = \varphi_k$ and $s_{t_k} = \phi_k$ within diagram (104) are correlated. Namely, we have

Theorem 35. *The evolution $\partial_{\varphi_k}(s)$ of the nonlocal variable s along the higher symmetry $\varphi_k = \Box(\phi_{k-1})$ of the Toda equation* (55) *coincides with the evolution ϕ_k that is defined by the k*th *higher analog $s_{t_k} = R_{\mathrm{pKdV}}^{k+1}(1)$ of the potential Korteweg–de Vries equation* (92):

$$\partial_{\Box(\phi_{k-1})}(s) = \phi_k = R_{\mathrm{pKdV}}(\phi_{k-1}).$$

Proof. From decompositions (103) it follows that

$$R_{\mathrm{pKdV}}(\phi_k) = D_x^{-1}\left(\partial_{\Box(\phi_k)}(T)\right).$$

$\Box$

Corollary 36. 1. Each shadow $\vec{\varphi}_k$ in the covering τ_s, see Eq. (72), over the Toda equation can be reconstructed up to the true nonlocal symmetry $\tilde{\partial}_{\vec{\varphi}_k,\phi_k}$ of the Toda equation.
2. All subdiagrams

$$
\begin{tikzcd}
\vec{\varphi}_{k+2} & \phi_{k+1} \arrow[l, "\Box"'] \\
\vec{\varphi}_{k+1} \arrow[u, "R_{\mathrm{Toda}}"] & \phi_k \arrow[l, "\Box"'] \arrow[u, "R_{\mathrm{pKdV}}"']
\end{tikzcd}
$$

in diagram (104) are commutative: we have $R_{\mathrm{Toda}}(\vec{\varphi}_{k+1}) = \Box(\phi_{k+1})$, and the relations

$$R_{\mathrm{Toda}} = \Box \circ R_{\mathrm{pKdV}} \circ \Box^{-1}, \qquad R_{\mathrm{pKdV}} = \Box^{-1} \circ R_{\mathrm{Toda}} \circ \Box$$

hold.

Remark 9. In the paper [40], the recursion operators R_{Toda} and R_{pKdV} were noted to be conjugate to each other in the scalar case (76-84).

Proof of the locality for the symmetry sequence $\mathfrak{A}$. All previous reasonings do not imply the locality of the elements ϕ_k of the sequence $\mathfrak{B}$ composed by higher symmetries of the potential Korteweg–de Vries equation (92). Indeed, these symmetries lie in the image of the operator D_x^{-1}. Therefore, the elements $\varphi_{k+1} = \square\,(\phi_k)$ of the sequence $\mathfrak{A}$ are not yet proved to be local, too.

Proposition 37 ([20]). *Recursion operator* (102) *for the potential Korteweg–de Vries equation* (92) *generates the* local in T *sequence of the higher symmetries*

$$\phi_k = R_{\mathrm{pKdV}}^{k+1}(\phi_{-1}) = \phi_k(T, \ldots, T_{2k}),$$

where $\phi_{-1} = 1$ *is the shift of the dependent variable* s *by a constant.*

There are several ways to prove Proposition 37. The method by I. S. Krasil'shchik ([61]) is based on the weak nonlocality of the recursion operator R_{pKdV} (see also Eq. (142) on page 85).

Corollary 38. The symmetries $\vec{\varphi}_k = \square(\phi_{k-1}) \in \mathfrak{A}$ of the Toda equation (55) are local and depend on the derivatives u^j_σ, $|\sigma| \geq 1$ for all $k \geq 0$.

We also note that in the paper [48] we analysed the case $r = 1$ and established the locality of the higher symmetries $\mathfrak{A} \subset \operatorname{sym} \mathcal{E}_{\mathrm{Liou}}$ directly, leaving apart the discussion upon the properties of the potential equation (92).

4.2. **Commutativity of the hierarchy** $\mathfrak{A}$**.** A classical example of evolution equations that admit a commutative symmetry subalgebra is given in the following lemma.

Lemma 39 ([36]). *Suppose* $\mathcal{E}$ *is the scalar evolution equation*

$$u_t = u_k + f(u_{k-2}, \ldots, u) \tag{105}$$

such that f *is a polynomial. Consider the Lie subalgebra*

$$\langle \varphi \in \operatorname{sym} \mathcal{E} \mid \varphi = \varphi(u_\sigma) \rangle \subseteq \operatorname{sym} \mathcal{E}$$

of its symmetries that depend on the variable u *and its derivatives only. Then this subalgebra is commutative.*

We denote by $\mathfrak{B}$ the minimal Lie subalgebra generated by the symmetries ϕ_k of the potential Korteweg–de Vries equation; here $k \geq -1$. From Lemma 39 it follows that the algebra $\mathfrak{B}$ is commutative:

$$\{\phi_k, \phi_l\} = \mathfrak{Э}_{\phi_k}(\phi_l) - \mathfrak{Э}_{\phi_l}(\phi_k) = 0,$$

therefore $\mathfrak{B}$ coincides with the linear span of its generators ϕ_k:

$$\mathfrak{B} = \operatorname{span}_{\mathbb{R}} \langle \phi_k \mid k \geq -1 \rangle.$$

Now we analyse the commutation properties of the symmetries $\varphi = \square(\phi(x, \mathbf{T}))$ of the Toda equations (55).

Lemma 40. *Suppose $\varphi' = \square\,(\phi'(x,\,\mathbf{T}))$ and $\varphi'' = \square\,(\phi''(x,\,\mathbf{T}))$. Then the Jacobi bracket $\{\varphi',\,\varphi''\}$, which was defined in Theorem 1, of the symmetries φ' and φ'' is*

$$\{\varphi', \varphi''\} = \square\,(\phi_{\{1,2\}}),$$

where the bracket

$$\phi_{\{1,2\}} = \partial_{\varphi'}(\phi'') - \partial_{\varphi''}(\phi') + \bar{D}_x(\phi')\,\phi'' - \phi'\,\bar{D}_x(\phi'') \tag{106}$$

on the arguments of the operator $\square$ is induced by the Jacobi bracket $\{\varphi', \varphi''\}$. Moreover,

$$\phi_{\{1,2\}} = \phi_{\{1,2\}}(x, \mathbf{T})$$

by Eq. (89).

Further on, we denote by $\mathfrak{A}$ the minimal Lie algebra generated by $\vec{\varphi}_k$ for all $k \geq 0$.

Theorem 41. *The Lie algebra $\mathfrak{A} \subset \operatorname{sym}\mathcal{E}_{\text{Toda}}$ is commutative*: $[\mathfrak{A}, \mathfrak{A}] = 0$*, and thence*

$$\mathfrak{A} = \operatorname{span}_{\mathbb{R}}\langle \vec{\varphi}_k \mid k \geq 0\rangle.$$

Proof. Commute two symmetries $\partial_{\varphi_{k_1}}$ and $\partial_{\varphi_{k_2}}$, apply the resulting evolutionary vector field to the variable s and take into account the relation between ϕ_k and φ_k. Hence we get

$$\begin{aligned}\left[\partial_{\varphi_{k_1}}, \partial_{\varphi_{k_2}}\right](s) = \partial_{\varphi_{k_1}}(\phi_{k_2}) - \partial_{\varphi_{k_2}}(\phi_{k_1}) = \\ \partial^{(s)}_{\phi_{k_1}}(\phi_{k_2}) - \partial^{(s)}_{\phi_{k_2}}(\phi_{k_1}) = \{\phi_{k_1},\, \phi_{k_2}\} = 0.\end{aligned}$$

Since $T = s_x$, thence

$$\left[\partial_{\varphi_{k_1}}, \partial_{\varphi_{k_2}}\right](T) = 0. \tag{107}$$

Now estimate the evolution of the integral T by using Lemma 40. First, consider the bracket $\{\varphi_{k_1},\, \varphi_{k_2}\}$ and then calculate $\dot{T}_{\phi_{\{k_1,k_2\}}}$. Recall that $\varphi_{k_1} = \square(\phi_{k_1-1})$ and $\varphi_{k_2} = \square(\phi_{k_2-1})$. Therefore,

$$\{\varphi_{k_1}, \varphi_{k_2}\} = \square(\phi_{\{k_1,k_2\}}),$$

where

$$\begin{aligned}\phi_{\{k_1,k_2\}} = \\ \partial_{\phi_{k_1}}(\phi_{k_2-1}) - \partial_{\phi_{k_2}}(\phi_{k_1-1}) + \bar{D}_x(\phi_{k_1-1})\,\phi_{k_2-1} - \phi_{k_1-1}\,\bar{D}_x(\phi_{k_2-1})\end{aligned}$$

owing to Eq. (106). From Example 15 it follows that

$$\begin{aligned}\partial_{\{\varphi_{k_1},\varphi_{k_2}\}}(T) = \partial_{\square(\phi_{\{k_1,k_2\}})}(T) = \\ = \left(-\beta\,\bar{D}_x^3 + T\,\bar{D}_x + \bar{D}_x \circ T\right)(\phi_{\{k_1,k_2\}}).\end{aligned} \tag{108}$$

Comparing Eq. (107) and Eq. (108), we get

$$\left(-\beta\,\bar{D}_x^3 + T\,\bar{D}_x + \bar{D}_x \circ T\right)\phi_{\{k_1,k_2\}} = 0. \tag{109}$$

In the left-hand side of Eq. (109) we obtain the operator $\hat{B}_2$ in total derivatives whose coefficients are elements of $\mathbf{T}$. We apply this operator to $\phi_{\{k_1,k_2\}}(T, \ldots, T_{\mu(k_1,k_2)})$ and get 0 in the right-hand side Thence, $\phi_{\{k_1,k_2\}} = 0$ and therefore

$$\{\varphi_{k_1}, \varphi_{k_2}\} = \Box(0) = 0.$$

The indexes k_1 and k_2 are arbitrary, therefore $\mathfrak{A}$ is commutative. $\Box$

Proposition 42. *Let $\mathcal{E}_{(0)} = \{u_{t_0} = \varphi_0(u_\sigma)\}$ be an evolution equation. Suppose that the symmetry $\varphi_k \in \operatorname{sym}\mathcal{E}_{(0)}$ is assigned to each $k \geq 0$ and assume that φ_k is independent of the time t_0 explicitly: $\varphi_k = \varphi_k(u_\sigma)$. Then the following two statements are equivalent:*

(1) *The algebra $\mathfrak{A} = \operatorname{span}_{\mathbb{R}}\langle\varphi_k \mid k \geq 0\rangle$ is a commutative Lie algebra: $\{\varphi_k, \varphi_l\} = 0$.*
(2) *The evolutionary vector field $\mathbf{\partial}_{\varphi_l}$ is a symmetry of the autonomous evolution equation $\mathcal{E}_{(k)} = \{u_{t_k} = \varphi_k\}$ for each k, $l \geq 0$.*

Proof. First, identify the evolutionary vector field $\mathbf{\partial}_{\varphi_k}$ with the autonomous evolution equation $u_{t_k} = \varphi_k$. Then, consider the equality

$$\mathbf{\partial}_{\varphi_k}(\varphi_l) = \mathbf{\partial}_{\varphi_l}(\varphi_k).$$

In its left-hand side we have

$$\mathbf{\partial}_{\varphi_k}(\varphi_l) = \mathbf{\partial}_{u_{t_k}}(\varphi_l) = \big(\bar{D}_{t_k} - \tfrac{\partial}{\partial t_k}\big)(\varphi_l) = \bar{D}_{t_k}(\varphi_l),$$

since φ_l does not depend on any time t_k. In the right-hand side of the equality we get

$$\mathbf{\partial}_{\varphi_l}(\varphi_k) = \ell_{\varphi_k}(\varphi_l).$$

Therefore, the comuutativity condition $\{\varphi_k, \varphi_l\} = 0$ for the symmetries φ_k and φ_l is equivalent to the determining equation

$$\big(\bar{D}_{t_k} - \ell_{\varphi_k}\big)(\varphi_l) = 0.$$

Thence φ_l is a symmetry of the equation $\mathcal{E}_{(k)}$ for any k, $l \geq 0$. $\Box$

Corollary 43. For any k, $l \geq 0$, the following propositions hold:

(1) The sections $\vec{\varphi}_k \in \mathfrak{A}$ are not only symmetries of the Toda equation, but also symmetries of all equations $\mathcal{E}_{(l)} = \{\boldsymbol{u}_{t_l} = \vec{\varphi}_l\}$:

$$\vec{\varphi}_k \in \operatorname{sym}\mathcal{E}_{(l)}^{\infty}.$$

(2) The recursion operator R_{Toda} is common for the whole tower of the evolution equations $\mathcal{E}_{(l)}$:

$$R_{\text{Toda}} \in \operatorname{Rec}\mathcal{E}_{(l)}.$$

In particular,

$$R_{\text{Toda}} = R_{\text{pmKdV}}.$$

Remark 10. In the paper [40], the scalar potential modified Korteweg–de Vries equation was considered in the gauge $u_t = u_3 + u_1^3$. In this case, this equation shares the recursion operator with the sine-Gordon equation $u_{xy} = \sin u$.

Remark 11. By using Corollary 38, we deduce that the section $\vec{\varphi}'_{-1} =$ const is a central extension of the commutative Lie subalgebra $\mathfrak{A} \subset \operatorname{sym}\mathcal{E}_{(l)}^{\infty}$ of symmetries of the evolution equation $\mathcal{E}_{(l)}$ for any $l \geq 0$, although const $\notin \operatorname{sym}\mathcal{E}_{\mathrm{Toda}}$.

Remark 12. By using formula (106), we establish a curious property of the Korteweg–de Vries equation (91). Recall that the proof of Theorem 41 is based on the equality $\phi_{\{k_1,k_2\}} = 0$ that holds for all elements ϕ_{k_1} and ϕ_{k_2} of the hierarchy $\mathfrak{B}$. Consider the first Hamiltonian structure $\hat{B}_1 = D_x$ for Eq. (91). We know that the Hamiltonians of the higher Korteweg–de Vries equations are in involution with respect to this structure. Thence we have

$$\big\langle \mathbf{E}_T(\mathcal{H}_i), \hat{B}_1 \circ \mathbf{E}_T(\mathcal{H}_j)\big\rangle = 0 \quad\Longleftrightarrow\quad \phi_i \cdot D_x(\phi_j) \in \operatorname{im} D_x.$$

Now we see that the last two summands in Eq. (106) are of this form. Therefore the sum

$$\mathbf{\ni}_{\phi_{k_1+1}}(\phi_{k_2}) - \mathbf{\ni}_{\phi_{k_2+1}}(\phi_{k_1}) \in \operatorname{im} D_x$$

is always a total derivative. The feature we observe is that each of these summands above is a total derivative by itself, therefore generalizing the involutivity property. Namely, by straighforward calculation we obtain Ω_{ij} such that the relations

$$\phi_i \cdot \frac{\partial}{\partial T}\big(D_x(\phi_j)\big) = D_x(\Omega_{ij})$$

hold for the elements of the hierarchy $\mathfrak{B}$. Several Ω_{ij} that correspond to i, $j \leq 2$, and the initial elements

$$\phi_{-1} = 1, \quad \phi_0 = s_1, \quad \phi_1 = -\beta s_3 + \tfrac{3}{2}s_1^2,$$
$$\phi_2 = \beta^2 s_5 - \tfrac{5}{2}\beta s_2^2 - 5\beta s_1 s_3 + \tfrac{5}{2}s_1^3$$

of the sequence $\mathfrak{B}$ are found in the table below:

	$j=1$	$j=2$
$i=-1$	$3T$	$-5\beta T_2+\frac{15}{2}T^2$
$i=0$	$\frac{3}{2}T^2$	$5T^3-5\beta TT_2+\frac{5}{2}\beta T_1^2$
$i=1$	$-\frac{3}{2}T_1^2+\frac{3}{2}T_2^2$	$\frac{5}{2}\beta^2T_2^2+\frac{35}{8}T^4-\frac{15}{2}\beta T^2T_2$
$i=2$	$-3\beta T_1T_3+\frac{3}{2}T_2^2+\frac{15}{8}T^4-\frac{15}{2}\beta TT_1^2$	$-\frac{5}{2}\beta^3T_3^2+\frac{15}{2}T^5+15\beta^2TT_1T_3-\frac{25}{2}\beta T^3T_2-\frac{5}{2}\beta^2T_1^2T_2+5\beta^2TT_2^2-\frac{75}{4}\beta T^2T_1^2.$

Obviously, $\Omega_{i,-1}=\Omega_{i,0}\equiv 0$ for all i.

5. The Hamiltonian formalism for the Euler equations

In this section, we consider the problem of constructing commutative Hamiltonian hierarchies of evolution equations associated with hyperbolic Euler systems (in particular, with the wave equation $s_{xy}=0$ or the Toda equation (55), see [51, 59, 60]. Also, we interpret the canonical coordinate–momenta formalism, which is widely used in mathematical physics (see [9, 77, 16] and references therein), within nonlocal Hamiltonian operators language of the jet-bundle framework ([10, 42, 63]) preserving the distinction between the coordinates and momenta. Next, we consider hyperbolic Euler equations (*i.e.*, we specify the ansatz for the Lagrangian density) and obtain the differential constraint between the dependent variables u^i and the momenta $\mathfrak{m}_j$. Then, we identify symmetries of these equations with autonomous potential evolution equations; also, we investigate the relation between Hamiltonian operators for potential and nonpotential evolution equations that describe the evolution of coordinates and momenta, respectively. The aim of our reasonings is to assign new jet bundle π' and the jet space $J^\infty(\pi')$ to the initial Euler equation $\mathcal{E}_{E-L}\subset J^k(\pi)$, see Eq. (15), such that the latter equation becomes an evolutionary vector field that belongs to $\varkappa(\pi')$, while the evolutionary fields that correspond to the momenta $\mathfrak{m}$ become elements of the adjoint module $\hat{\varkappa}(\pi')$. Finally, we relate commutative Lie subalgebras of the Noether symmetries for hyperbolic Euler equations and pairs of Hamiltonian hierarchies composed by potential and nonpotential evolution equations.

5.1. **Canonical formalism.** Consider an abstract $2r$-dimensional dynamical system with r dependent variables u^i, momenta $\mathfrak{m}_j$, the spatial coordinates x, and the time t, defined by the Poisson brackets ([9, 77])

$$\begin{gathered}\{u^i, u^j\}_{\bar{A}} = 0,\\ \{\mathfrak{m}_i, \mathfrak{m}_j\}_{\bar{A}} = 0,\\ \{u^i(x,t), \mathfrak{m}_j(x',t)\}_{\bar{A}} = \bar{A}^i_j\,\delta(x-x');\end{gathered} \tag{110}$$

where $\bar{A}$ is an $(r\times r)$-matrix differential operator in total derivatives

$$D_\sigma = (D_{x^1})^{\sigma_1}\cdots(D_{x^n})^{\sigma_n}$$

with respect to x ($\sigma = (\sigma_1, \ldots, \sigma_n)$ is a multiindex) and the skew-symmetric bracket acts as the derivation with respect to any of its arguments. Assume that $\mathcal{H} = [H\,\mathrm{d}x]$ is a Hamiltonian with the density

$$H(x) = H(u(x), u_\sigma(x); \mathfrak{m}(x), D_\sigma\mathfrak{m}(x)).$$

Then the dynamics

$$\begin{aligned}\dot{u} &= \bigl\{u(x), H\bigl(u(x'), \mathfrak{m}(x')\bigr)\bigr\}_{\bar{A}},\\ \dot{\mathfrak{m}} &= \bigl\{\mathfrak{m}(x), H\bigl(u(x'), \mathfrak{m}(x')\bigr)\bigr\}_{\bar{A}}\end{aligned} \tag{111}$$

of the variables u and $\mathfrak{m}$ is obtained in a standard way:

$$\dot{u} = \{u(x), H(x')\}_{\bar{A}} = \oint_{C(x)}\sum_\sigma\{u(x), D_\sigma\mathfrak{m}(x')\}_{\bar{A}}\cdot\frac{\partial H}{\partial\,(D_\sigma\mathfrak{m}(x'))}\,\mathrm{d}x'.$$

Here $C(x)$ is a small contour around the point x. We emphasize that this language of local coordinates and the δ-distribution is spread in many works on the Hamiltonian formalism application in the field theory; in the sequel, we pass to the invariant exposition soon. Meanwhile, integrating by parts we obtain

$$\dot{u} = \bar{A}\circ\frac{\delta H}{\delta\mathfrak{m}(x)}. \tag{112a}$$

Similarly we obtain the second relation

$$\dot{\mathfrak{m}} = -\bar{A}\circ\frac{\delta H}{\delta u(x)}. \tag{112b}$$

These reasonings motivate the following definition of the variational bracket of two Hamiltonians with the densities H and H', respectively: we set

$$\{H, H'\}_{\bar{A}} = \frac{\delta H}{\delta u}\cdot\bar{A}\,\frac{\delta H'}{\delta\mathfrak{m}} - \frac{\delta H}{\delta\mathfrak{m}}\cdot\bar{A}\,\frac{\delta H'}{\delta u}.$$

Remark 13. Usually, all dependent variables are treated uniformly. In our case, the momenta can be absorbed by the additional variables $u^{r+j} = \mathfrak{m}_j$ for $1\leq j\leq r$. In other words, we enlarge the total number $m = 2r$ of the coordinates u^j twice. The operators $\bar{A}$ and $-\bar{A}$ are also united in the $(m\times m)$-matrix

$$A = \begin{pmatrix}0 & \bar{A}\\ \bar{A}^* & 0\end{pmatrix},$$

such that the dynamical equations take the form

$$\dot{\vec{u}} = A \circ \frac{\delta\bigl(H(\boldsymbol{u})\bigr)}{\delta\vec{u}},$$

where the variation $\delta/\delta\vec{u}$ is calculated with respect to the new vector $\vec{u} \equiv {}^t(u, \mathfrak{m})$. Due to this reason, Remark 13 motivates definitions 6–8, see page 10 in the Introduction.

A traditional approach ([10, 42, 63]) to description of Hamitonian PDE dynamics does not appeal to any distinction between the dependent variables, coordinates u and momenta $\mathfrak{m}$. Nevertheless, by using the double set of $m = 2r$ dependent varuables u and $\mathfrak{m}$ we describe several remarkable properties of models of mathematical physics, for example, of the Korteweg–de Vries equation (91), the modified Korteweg–de Vries equation (84), and similar equations. These aspects are discussed in Sec. 7 below.

6. Hyperbolic Euler equations

Consider a first-order Lagrangian

$$\mathcal{L} = \int L(u, u_x, u_y; x, y)\,\mathrm{d}x \wedge \mathrm{d}y$$

with the density

$$L = -\tfrac{1}{2}\sum_{i,j} \bar{\kappa}_{ij} u_x^i u_y^j + H(u; x, y),$$

where $\bar{\kappa}$ is a nondegenerate, symmetric, constant $(r \times r)$-matrix. We note that the notation $\bar{\kappa}$ is correlated with the general exposition. Suppose $r = 1$ and $H = 0$, then we obtain the wave equation (75) (see Sec. 3). If $\bar{\kappa} = \kappa = \|a_i k_{ij}\|$ and the function H is defined by formula (117), then we get the Toda equations.

Choose the independent variable y for the "time" coordinate (the nondegeneracy condition implies $\partial L/\partial u_y \neq 0$), leave x for the spatial coordinate on the base $\mathbb{R}$ of the new jet bundle $\pi' \colon \mathbb{R} \times \mathbb{R} \xrightarrow{u} \mathbb{R}$ with the old fiber coordinate u, and denote by $\mathfrak{m}_j = \partial L/\partial u_y^j$ the jth conjugate coordinate (momentum) for the jth dependent variable u^j for any $1 \leq j \leq r$:

$$\mathfrak{m}_i = -\tfrac{1}{2}\sum_{j=1}^{r} \bar{\kappa}_{ij} u_x^j. \tag{113}$$

The differential constraint, Eq. (113), between coordinates and momenta for the initial equation (15) is our main tool that allows to construct the Hamiltonian structures.

Consider the Legendre transform

$$H\,\mathrm{d}x \wedge \mathrm{d}y = \left\langle \mathfrak{m}, \frac{\partial L}{\partial \boldsymbol{u}_y} \right\rangle - \mathcal{L}$$

and assign the Hamiltonian

$$\mathcal{H}(u,\mathfrak{m}) = [H\,\mathrm{d}x]$$

to the Lagrangian $\mathcal{L}$. We decompose its density H into the sum of two equal summands and use the relation

$$u = -2\bar{\kappa}^{-1}\,\bar{D}_x^{-1}(\mathfrak{m})$$

in one of the components, in agreement with (112):

$$H = \tfrac{1}{2}H[u] + \tfrac{1}{2}H[\mathfrak{m}].$$

The hyperbolic Euler equation

$$\mathcal{E}_{E-L} = \{\mathbf{E}_u(\mathcal{L}) = 0\}$$

is equivalent to the system

$$u_y = \frac{\delta H}{\delta \mathfrak{m}}, \qquad \mathfrak{m}_y = -\frac{\delta H}{\delta u} \tag{114}$$

with respect to the *canonical* Hamiltonian structure $\bar{A} = \mathbf{1}$. Owing to the relations

$$\frac{1}{2}\frac{\delta}{\delta\mathfrak{m}} = \underbrace{\bar{\kappa}^{-1}\cdot D_x^{-1}}_{A_1}\circ\frac{\delta}{\delta u}, \qquad \frac{\delta}{\delta u} = \frac{1}{2}\underbrace{D_x\circ\bar{\kappa}}_{\hat{A}_1}\cdot\frac{\delta}{\delta\mathfrak{m}},$$

the dynamical equations are separated:

$$\begin{aligned} u_y &= A_1\circ\mathbf{E}_u\bigl([H[u]\,\mathrm{d}x]\bigr),\\ \mathfrak{m}_y &= -\tfrac{1}{2}\hat{A}_1\circ\mathbf{E}_{\mathfrak{m}}\bigl([H[\mathfrak{m}]\,\mathrm{d}x]\bigr). \end{aligned} \tag{115}$$

Example 18. Suppose that $r = 1$, $\bar{\kappa} = \|1\|$, and the Hamiltonian is trivial: $H \equiv 0$. Then the Hamiltonian operators

$$B_1 = D_x^{-1}, \qquad \hat{B}_1 = D_x$$

assigned to wave equation (75) are mutually inverse.

Now we apply these reasonings to the Toda equations (55), which are assigned to Lagrangian (56). It turns out that their Hamiltonian representation (118) hints the existence of the minimal integral T, see Eq. (58). The conservation of T is induced by the conservation of Hamiltonian density (117) for Eq. (55).

We start with a general

Lemma 44. *Assume that the density H of a Hamiltonian $\mathcal{H} = [H\,\mathrm{d}x]$ for the Hamiltonian evolution equation $u_t = A(\mathbf{E}_u(\mathcal{H}))$ does not depend on the time t explicitly. Then the density H is conserved for this equation*:

$$\bigl[\bar{D}_t(H)\,\mathrm{d}x\bigr] = 0.$$

Proof. By using the condition $\partial H/\partial t = 0$, we calculate the derivative $\bar{D}_t(H)$:

$$\bar{D}_t(\mathcal{H}) = \big\langle 1, \ni_{A\circ\mathbf{E}_u(\mathcal{H})}(H)\big\rangle = \sum\nolimits_\sigma \Big\langle \frac{\partial H}{\partial u_\sigma}, \bar{D}_\sigma\big(A\circ\mathbf{E}_u(\mathcal{H})\big)\Big\rangle =$$

integrating by parts, we obtain

$$= \sum\nolimits_\sigma \big\langle (-1)^\sigma\, \bar{D}_\sigma\Big(\frac{\partial H}{\partial u_\sigma}\Big), A\circ\mathbf{E}_u(\mathcal{H})\big\rangle =$$

since the Hamiltonian operator A is skew–symmetric, $A^* = -A$, we get

$$= -\big\langle A\circ\mathbf{E}_u(\mathcal{H}), \mathbf{E}_u(\mathcal{H})\big\rangle.$$

Again, by using the definition of the Euler operator $\mathbf{E}_u$ and integrating by parts, we have

$$\begin{aligned} &= -\Big\langle \sum\nolimits_\sigma \bar{D}_\sigma\big(A\circ\mathbf{E}_u(\mathcal{H})\big), \frac{\partial H}{\partial u_\sigma}\Big\rangle = \\ &= -\big\langle \ni_{A\circ\mathbf{E}_u(H)}(H), 1\big\rangle = -\bar{D}_t(\mathcal{H}). \end{aligned}$$

Moving the result to the left-hand side of the initial equality, we finally obtain the required condition $2\bar{D}_t(H) \in \operatorname{im}\bar{D}_x$. □

Example 19 ([59]). Choose the coordinate y for the "time", define the momenta (see (113)) $\mathfrak{m} = \partial L/\partial \boldsymbol{u}_y$:

$$\mathfrak{m}_i = \tfrac{1}{2}\sum_{j=1}^{r} \kappa_{ij} u^j, \tag{116}$$

and obtain the density H_{Toda} of the Hamiltonian $\mathcal{H}_{\mathrm{Toda}}$:

$$H_{\mathrm{Toda}}(u,\mathfrak{m}) = -\tfrac{1}{2}\sum_{i=1}^{r} a_i\, \exp\left(\frac{2}{a_i}\, D_x^{-1}(\mathfrak{m}_i)\right) - \tfrac{1}{2}\sum_{i=1}^{r} a_i\, \exp\Big(\sum_{j=1}^{r} k_{ij}u^j\Big).$$

We have already decomposed the density H into the sum of two components that depend on $\mathfrak{m}_i$ and u^j, respectively. Then, the canonical Hamiltonian representation of the Toda equation $\mathcal{E}_{\mathrm{Toda}}$ is

$$\begin{cases} \dot{u}^i = \dfrac{\delta H_{\mathrm{Toda}}}{\delta \mathfrak{m}_i} = D_x^{-1}\Big(\exp\Big(\displaystyle\sum_{j=1}^{r} k_{ij}u^j\Big)\Big), \\ \dot{\mathfrak{m}}_i = -\dfrac{\delta H_{\mathrm{Toda}}}{\delta u^i} = -\dfrac{1}{2}\displaystyle\sum_{j=1}^{r} \kappa_{ij}\, \exp\Big(\sum_{l=1}^{r} k_{jl}u^l\Big). \end{cases}$$

Therefore, in terms of the dependent variables $\boldsymbol{u}$, we have

$$H_{\mathrm{Toda}}(\boldsymbol{u}) = \sum_{i=1}^{r} a_i\, \exp\Big(\sum_{j=1}^{r} k_{ij}u^j\Big) \tag{117}$$

and

$$\dot{\boldsymbol{u}} = A_1 \circ \mathbf{E}_{\boldsymbol{u}}\bigl(\mathcal{H}_{\text{Toda}}(\boldsymbol{u})\bigr), \tag{118}$$

where $A_1 = \hat{\kappa}^{-1} \cdot D_x^{-1}$. Recall that an evolution representation for the Liouville equation (76) was obtained earlier in Eq. (86).

In order to correlate the resulting expressions with (58), we apply the transformation $x \leftrightarrow y$ to the Hamiltonian equation (118), consider the Hamiltonian density (117), which is independent of y, and use Lemma 44. Then we obtain

$$\bar{D}_x(H_{\text{Toda}}) = \bar{D}_y\Bigl(\sum_{i=1}^{r} a_i u^i_{xx}\Bigr) = \bar{D}_y\Bigl(\tfrac{1}{2}\sum_{i,j=1}^{r} \kappa_{ij} u^i_x u^j_x\Bigr), \tag{119}$$

whence follows expression (58).

By using the Hamiltonian representation (118), we describe the negative elements φ_k of the sequence $\mathfrak{A}$ with $k < 0$:

Proposition 45. *The right-hand side of the Hamiltonian evolution equation*

$$u_y = \hat{\kappa}^{-1} \circ D_x^{-1} \circ \mathbf{E}_u(\mathcal{H}_{\text{Toda}}) \tag{118$'$}$$

is the inverse image of the translation $u_{t_0} = \varphi_0$ *with respect to the mapping* R_{Toda}; *the translation* u_y *is the element* φ_{-1} *of the sequence* $\mathfrak{A}$.

Proof. By Lemma 34, we have

$$R_{\text{Toda}}(\vec{\varphi}_{-1}) = \square\left(\ni_{\boldsymbol{u}_y}(s)\right) = \square(\bar{D}_y(s)) = \square(1) = u_x.$$

$\square$

Therefore, representations (118) and (118$'$) of the Toda equation provide the translations

$$u_y = \varphi_{-1} = \hat{\kappa}^{-1} \circ \bar{D}_x^{-1} \circ \mathbf{E}_u(H_{\text{Toda}})$$

and

$$u_x = \varphi_0 = \hat{\kappa}^{-1} \circ \bar{D}_y^{-1} \circ \mathbf{E}_u(H_{\text{Toda}})$$

that are related by the recursion operator R_{Toda}:

$$\varphi_{-2} = \bar{\square}(\bar{T}) \xleftarrow[R'_{\text{Toda}}]{} u_y = \varphi_{-1} \underset{R'_{\text{Toda}}}{\overset{R_{\text{Toda}}}{\rightleftarrows}} \varphi_0 = u_x \xrightarrow{R_{\text{Toda}}} \varphi_1 = \square(T)$$

The operator R'_{Toda} obtained from R_{Toda} by using the discrete symmetry $x \leftrightarrow y$ generates the symmetries $\varphi_k \in \mathfrak{A}$ of the Toda equation with $k \leq -1$.

Now we return to the discussion on the Hamiltonian formalism for the Euler equations $\mathcal{E}_{E-L} = \{\mathbf{E}_u(\mathcal{L}) = 0\}$. We identify the symmetries φ of these equations with the evolution equations $u_t = \varphi$. Consider a symmetry $\varphi(u_x,\, u_{xx},\, \ldots)$ of Eq. (15), *i.e.*, the *potential* evolution equation

$$u_t = \varphi(u_x, u_{xx}\ldots), \tag{120a}$$

then the induced evolution $\mathfrak{m}_t$ of the momenta is described by the *non*-potential equation

$$\mathfrak{m}_t = -\tfrac{1}{2}\bar{\kappa} \cdot D_x\bigl({}^t\varphi(\mathfrak{m}, \mathfrak{m}_x, \ldots)\bigr). \tag{120b}$$

In addition, assume that the evolution φ is Hamiltonian:

$$\dot{u} = \frac{1}{2}\frac{\delta H}{\delta \mathfrak{m}}, \qquad \dot{\mathfrak{m}} = -\frac{1}{2}\frac{\delta H}{\delta u}.$$

Then we have

$$\begin{aligned} \dot{u} &= \underbrace{\bar{\kappa}^{-1} \cdot D_x^{-1}}_{A_1} \frac{\delta H}{\delta u}, \\ \dot{\mathfrak{m}} &= -\tfrac{1}{2} \underbrace{D_x \cdot \bar{\kappa}}_{\hat{A}_1} \frac{\delta H}{\delta \mathfrak{m}}, \end{aligned} \tag{121}$$

i.e., *both* equations (120) are Hamitonian simultaneously and their Hamiltonian structures A_1 and $\hat{A}_1$ are mutually inverse.

We note two classical examples of the pairs of evolution equations that admit mutually inverse Hamiltonian operators.

Example 20 ([59]). The potential KdV equation (92) is Hamiltonian with respect to the operator $B_1 = D_x^{-1}$, while $\hat{B}_1 = D_x$ is the first Hamiltonian operator (see (10)) for the KdV equation (91). One can easily verify that Eq. (92) is compatible with the wave equation (75), *i.e.*, the right-hand side ϕ_1 in the potential KdV equation $s_t = \phi_1$ is a symmetry of $s_{xy} = 0$, while the nonpotential equation (91) describes the evolution of the momentum $T = s_x$ (up to the constant factor $-1/2$).

The potential and nonpotential modified Korteweg–de Vries equations, Eq. (93) and (127), respectively, supply another example. In fact, the succeeding subsections 7.1 and 7.2 contain the detailed analysis of these two examples. We demonstrate that the first pair defines the symmetry hierarchy $\mathfrak{B}$ for wave equation (75) and the second pair is related with the hierarchy $\mathfrak{A}$ of symmetries of the Toda equation (55).

Now we return to Eq. (121) and note an important property of the Hamiltonian equations

$$\dot{u} = A_1 \circ \mathbf{E}_u(\mathcal{H})$$

and

$$\dot{\mathfrak{m}} = -\tfrac{1}{2}\hat{A}_1 \circ \mathbf{E}_{\mathfrak{m}}(\mathcal{H}) = -\tfrac{1}{2}\mathbf{E}_u(\mathcal{H}).$$

It turns out that we have already met the expression in the right-hand side of the latter equation in Lemma 5 on page 13. Lemma 5 assigns the generating sections $\psi_\eta = \mathbf{E}_u(\eta_0)$ to the densities η_0 of conservation laws $[\eta]$ for evolution equations. Formulas (120) give an interpretation of Lemma 5: the generating sections ψ describe (up to the sign) the

evolution $\dot{\mathfrak{m}}$ of the momenta $\mathfrak{m}$ along the Hamiltonian symmetries φ of the initial equation $\mathcal{E}$.

Moreover, we relate two pairs of mappings of different types. Assume that there are recursion operators R_u and $R_{\mathfrak{m}}$ for *different* evolution equations (120a) and (120b), respectively, and consider the mappings

$$R_u\colon \varkappa \to \varkappa, \qquad \mathcal{T}_u\colon \hat{\varkappa} \to \hat{\varkappa}$$

which produce symmetries and generating sections of conservation laws for the same equation $\mathcal{E}_u$, respectively. The relation ([42])

$$\mathcal{T}_u = R_u^* \tag{122}$$

holds for evolution equations; see, for example, diagram (141) on page 84. In terms of the (120), Eq. (122) means that the skew–symmetric recursion operator

$$A_R = \begin{pmatrix} 0 & R_u \\ -R_u^* & 0 \end{pmatrix} \tag{123}$$

is defined for the Hamiltonian representation (115) of the initial Euler equation $\mathcal{E} = \{\mathbf{E}_u(\mathcal{L}) = 0\}$. This situation is realized for the above–mentioned pairs of equations, see (130) on page 73. We also note that the condition $R_v = R_u^*$ is valid in this case.

Now we study the problem of constructing a bi-Hamiltonian hierarchy (to be more precise, a pair of hierarchies with respect to u and $\mathfrak{m}$) by using the recursion operator (123). Namely, we find the conditions for the operator A_R to induce the *second* structure $\{u, \mathfrak{m}\}_{A_R}$ on the Hamiltonians $\mathcal{H}_i$ in the diagram

$$\begin{tikzcd}
\mathcal{H}_0 \arrow[d, "\frac{\delta}{\delta\mathfrak{m}}"'] & \mathcal{H}_1 \arrow[d, "\frac{\delta}{\delta\mathfrak{m}}"'] & \mathcal{H}_2 \arrow[d, "\frac{\delta}{\delta\mathfrak{m}}"'] & \\
\varphi_0 \arrow[r, "R"'] & \varphi_1 \arrow[r, "R"'] & \varphi_2 \arrow[r, "R"'] & \cdots
\end{tikzcd}$$

such that the Jacobi identity (7b) holds. The commutativity condition $[\varphi_i, \varphi_j] = 0$ is sufficient for the Jacobi identity to hold for bracket (6), supplied by the operator A_R; the sections $\mathfrak{m}_{t_i} = \psi_i$ also commute in this case. We denote by $\mathfrak{U}$ the minimal Lie algebra generated by the sections $u_{t_i} = \varphi_i$. We emphasize that, in general, we require these operators (123) to provide the Lie algebra structure (7b) on the Hamiltonians $\mathcal{H}_i \in \bar{H}^n(\pi)$ but not on the whole horizontal cohomology group $\bar{H}^n(\pi)$. Therefore, our concept extends Definition 7 of a Hamiltonian operator since we treat the particular case of a hyperbolic Euler equation $\mathcal{E}$ and a sequence of its Hamiltonian symmetries φ_i whose Hamiltonians are $\mathcal{H}_i$, respectively. Further on, suppose $\mathfrak{U}$ is a Lie subalgebra of the *Noether* symmetries of the Lagrangian $\mathcal{L}$. This assumption is sufficient for the existence of the Hamiltonians $\mathcal{H}_i$ such that

$$\varphi_i = R\,\frac{\delta\mathcal{H}_{i-1}}{\delta\mathfrak{m}} = \mathbf{1}\cdot\frac{\delta\mathcal{H}_i}{\delta\mathfrak{m}}, \qquad \psi_i = -R^*\,\frac{\delta\mathcal{H}_{i-1}}{\delta u} = -\mathbf{1}\cdot\frac{\delta\mathcal{H}_i}{\delta u}. \tag{124}$$

Indeed, the existence of the conserved densities is prescribed by the Noether theorem (see Theorem 6 on page 14).

We obtain a more usual description of the pair of the Magri schemes (see [70]) by using the splitting (121):

$$
\begin{tikzcd}
 & \psi_{i+1} \arrow[dr, "A_1"] & \cdots & \psi_{i+1} & \cdots & \\
 & & \varphi_{i+1} \arrow[u, "R"'] & & \varphi_i \arrow[ul, "\hat{A}_2"'] \arrow[u, "R"'] \arrow[dl, "\hat{A}_1"] & \mathcal{H}_i \arrow[l, "\mathbf{E}_{\mathfrak{m}}"'] \\
\mathcal{H}_i \arrow[r, "\mathbf{E}_u"] & \psi_i \arrow[uu, "R^*"] \arrow[ur, "A_2"] \arrow[dr, "A_1"] & & \psi_i \arrow[uu, "R^*"] & & \\
 & & \varphi_i \arrow[uu, "R,"'] & & \varphi_{i-1} \arrow[uu, "R"'] \arrow[ul, "\hat{A}_2"'] \arrow[dl, "\hat{A}_1"] & \mathcal{H}_{i-1}. \arrow[l, "\mathbf{E}_{\mathfrak{m}}"'] \\
 & \psi_{i-1} \arrow[uu, "R^*"] \arrow[ur, "A_2"] & \cdots \arrow[u, "R"'] & \psi_{i-1} \arrow[uu, "R^*"] & \cdots \arrow[u, "R"'] &
\end{tikzcd}
\tag{125}
$$

Here the operators A_1 and $\hat{A}_1$ are defined by constraint (113) between u and $\mathfrak{m}$, while the second operators A_2 and $\hat{A}_2$ originate from the relations

$$R = A_2 \circ A_1^{-1}, \quad R^* = \hat{A}_2 \circ \hat{A}_1^{-1}$$

respectively. In the sequel, we preserve the notation and mark the operators $\hat{A}_{1,2}$ and $\hat{B}_{1,2}$ for the nonpotential equations (127) and (91), respectively, with the 'hat' sign.

The notion of the ℓ^*-*covering* in the category of differential equations was introduced in [42], and the unifying approach towards the recursion operators $R\colon \varkappa \to \varkappa$, the conjugate recursion operators $\mathcal{T}\colon \hat{\varkappa} \to \hat{\varkappa}$, the Hamiltonian structures $A\colon \hat{\varkappa} \to \varkappa$, and the symplectic structures $\hat{A}\colon \varkappa \to \hat{\varkappa}$ was elaborated for the Magri scheme (125) technique. The return from diagram (125) to (124) provides new interpretation of these structures and their interrelations.

Remark 14. The correlation between the generating sections of conservation laws for the initial Euler equation $\mathcal{E}_{E-L} = \{F \equiv \bar{\kappa} \cdot \mathbf{E}_u(\mathcal{L}) = 0\}$ (these sections are denoted by $\psi_{\mathcal{L}}$) and for the same equation in its evolution representation (115) (we denote these sections by ψ) is given by the diagram

$$\psi \xrightarrow{-D_x^{-1}} \psi_{\mathcal{L}} \xrightarrow{\bar{\kappa}^{-1}} \varphi, \tag{126}$$

where the first arrow follows from the definition of a generating section:

$$\mathrm{d}_h \eta = \nabla(F)\,\mathrm{d}\boldsymbol{x} = \nabla \circ D_x(D_x^{-1}(F))\,\mathrm{d}\boldsymbol{x},$$

and, therefore,

$$\psi_{\mathcal{L}} = \nabla^*(1), \quad \psi = -D_x \circ \nabla^*(1),$$

while the second arrow is induced by Lemma 9.

7. Properties of the Korteweg–de Vries hierarchies

In this section, we illustrate the concept ([77, 59, 60]) formulated in the preceding section. In what follows, we consider the bi–Hamiltonian equations (91), (92), (93), and the r-component analogs

$$\mathcal{E}_{\mathrm{mKdV}} = \{\vartheta_{t_1} = D_x \circ \hat{\kappa} \circ \square(T)\} \tag{127}$$

of the modified Korteweg–de Vries equation (84). We also introduce the new variable

$$\vartheta = \kappa \cdot u_x \tag{128}$$

that is subject to the latter equation. This dependent variable differs from the momenta

$$\mathfrak{m}_{\mathrm{Toda}} = -\tfrac{1}{2}(\kappa\, u_x)^*$$

by the constant factor (-2). This gauge simplifies the calculations and resulting expressions.

Now we study the relations between the equations $\mathcal{E}_{\mathrm{pKdV}}$ and $\mathcal{E}_{\mathrm{KdV}}$, and between $\mathcal{E}_{\mathrm{pmKdV}}$ and $\mathcal{E}_{\mathrm{mKdV}}$, respectively. We claim that the potentials u and s for ϑ and T are such that the linearizations

$$\ell_\vartheta^{(u)} = D_x \circ \hat{\kappa}, \quad \ell_T^{(s)} = D_x$$

are equal to the first Hamiltonian structures $\hat{A}_1$ and $\hat{B}_1$ for the equations $\mathcal{E}_{\mathrm{mKdV}}$ and $\mathcal{E}_{\mathrm{KdV}}$, respectively. The first Hamiltonian structures A_1 and B_1 for $\mathcal{E}_{\mathrm{p(m)KdV}}$ are inverse with respect to $\hat{A}_1$ and $\hat{B}_1$. The Hamiltonians for the equations $\mathcal{E}_{\mathrm{(p)KdV}}$ are well-known. The Hamiltonians $\mathcal{H}_k$ for the equations $\mathcal{E}_{\mathrm{(p)mKdV}}$ are described in final Theorem 52 on page 79.

7.1. **The Korteweg–de Vries equation.** Let $\mathcal{L}_s$ be the Lagrangian

$$\mathcal{L}_s = -\bigl[\tfrac{1}{2} s_x s_y \, \mathrm{d}x \wedge \mathrm{d}y\bigr].$$

Assign the wave equation

$$\mathcal{E}_s = \{s_{xy} = 0\},$$

see Eq. (75) on page 46, to $\mathcal{L}_s$ and consider the commutative Lie subalgebra

$$\mathfrak{B} = \mathrm{span}_{\mathbb{R}}\langle R^{k+1}_{\mathrm{pKdV}}(\phi_{-1}),\ \phi_{-1} = 1,\ k \geq 0\rangle \subset \mathrm{sym}\,\mathcal{E}_s$$

of its symmetries. We identify this subalgebra with the hierarchy of the higher potential Korteweg–de Vries evolution equations, see Eq. (92). We continue using the variable $T = s_x$, whose evolution is described by the elements of the *non*potential Korteweg–de Vries hierarchy for Eq. (91), instead of the canonical momentum $\mathfrak{m}_s = -\frac{1}{2}s_x$ for the wave equation. By (121), these two hierarchies admit the pair of mutually inverse Hamiltonian operators $\hat{B}_1 = B_1^{-1} = D_x$ for the equations $\mathcal{E}_{\mathrm{KdV}}$ and $\mathcal{E}_{\mathrm{pKdV}}$, respectively. The coinciding elements within the initial part

of the Magri schemes (see Eq. (125)) for $\mathcal{E}_{\text{KdV}}$ and $\mathcal{E}_{\text{pKdV}}$ are underlined in the diagram below:

$$
\begin{tikzcd}
 & & \dots & \\
h_1[s] \arrow[r, "\mathbf{E}_s"] \arrow[dr, "-\operatorname{id}"'] & \underline{\underline{-\beta s_4 + 3 s_1 s_2}} \arrow[ur, "B_2"] \arrow[dr, "B_1 = D_x^{-1}"] & & \dots \\
 & h_1[T] = \left[\tfrac{1}{2}(\beta T_1^2 + T^3)\,\mathrm{d}x\right] \arrow[r, "\mathbf{E}_T"] & -\beta s_3 + \tfrac{3}{2} s_1^2 \arrow[uu, "R_{\text{pKdV}}"] \arrow[ur, "\hat{B}_2"] \arrow[dr, "\hat{B}_1 = D_x"] & \\
h_0[s] \arrow[r, "\mathbf{E}_s"] \arrow[dr, "-\operatorname{id}"'] & \underline{s_2} \arrow[ur, "B_2"] \arrow[dr, "B_1"] & & \underline{\underline{-\beta T_3 + 3T \cdot T_1}} \arrow[uu, "R_{\text{KdV}}"'] \\
 & h_0[T] = \left[\tfrac{1}{2} T^2\,\mathrm{d}x\right] \arrow[r, "\mathbf{E}_T"] & s_1 \arrow[uu, "R_{\text{pKdV}}"] \arrow[ur, "\hat{B}_2"] \arrow[dr, "\hat{B}_1"] & \\
 & & & \underline{T_1} \arrow[uu, "R_{\text{KdV}}"'] \\
 & h_{-1}[T] = [T\,\mathrm{d}x] \arrow[r, "\mathbf{E}_T"] & 1 \arrow[uu, "R_{\text{pKdV}} = R^*_{\text{KdV}}"] \arrow[ur, "\hat{B}_2 = -\beta D_x^3 + D_x \circ T + T \cdot D_x"'] &
\end{tikzcd}
\tag{129}
$$

The initial terms of this diagram are given by the Hamiltonians whose densities are known from the vastest literature:

$$
h_{-1} = [T\,\mathrm{d}x], \qquad h_0 = [\tfrac{1}{2} T^2\,\mathrm{d}x], \qquad h_1 = [\tfrac{1}{2}(\beta T_x^2 + T^3)\,\mathrm{d}x], \quad \text{etc.}
$$

Now we consider an example to the relation (122). Namely, the identity

$$
R_{\text{pKdV}} = R^*_{\text{KdV}}
$$

is satisfied by the recursion operator R_{pKdV} for the potential Korteweg–de Vries equation (92) and the recursion operator

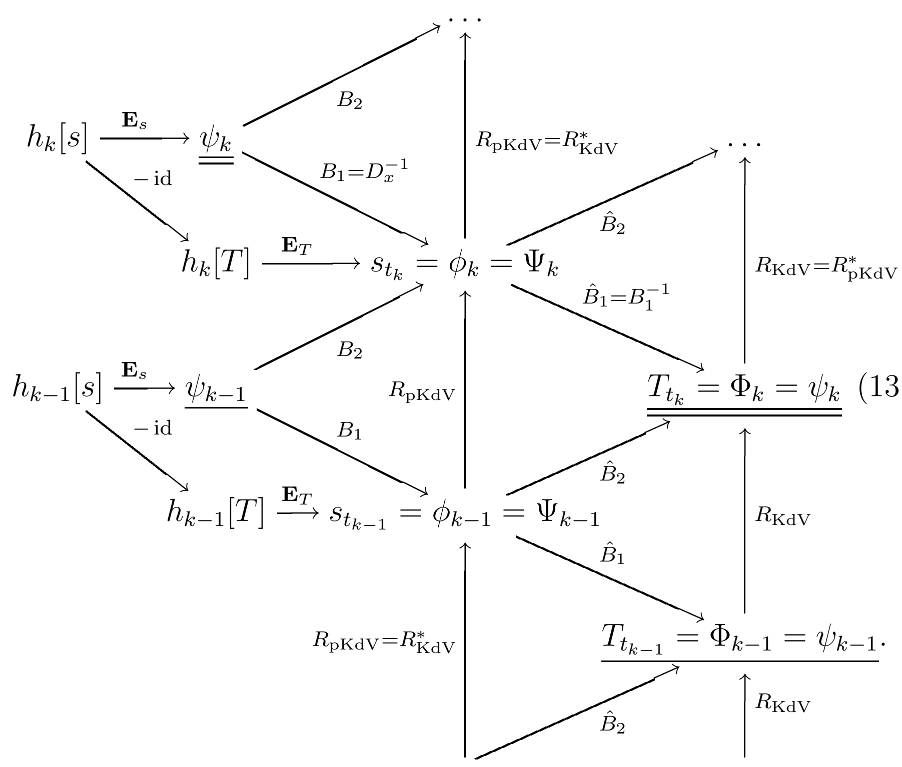
$$
R_{\text{KdV}} = -\beta D_x^2 + 2T + T_1 \cdot D_x^{-1}
$$

for the Korteweg–de Vries equation (91). The recursion R_{pKdV} is factorized by the Hamiltonian structures as follows:

$$
\begin{aligned}
\overbrace{(-\beta\, D_x + s_1 \cdot D_x^{-1} + D_x^{-1} \circ s_1)}^{B_2} \circ D_x &= \\
& D_x^{-1} \circ \underbrace{(-\beta\, D_x^3 + D_x \circ T + T \cdot D_x)}_{\hat{B}_2}.
\end{aligned}
\tag{130}
$$

The second Hamiltonian structure $\hat{B}_2$ for Eq. (91) is a factor in the right-hand side of Eq. (130). Recall that this structure appeared also in Remark 15 on page 50. Another important property of this structure is the correlation between the equation $\mathcal{E}_{\mathrm{KdV}}$, the Toda equation $\mathcal{E}_{\mathrm{Toda}}$, and the Virasoro algebra ([39]). We analyse the latter property in Remark 17, see the end of this subsection.

The Hamiltonian operators B_1 and $\hat{B}_1$ are inverse, therefore the Magri schemes (125) for Eq. (92) and (91) are correlated.



(131)

Here ϕ_k and Φ_k are symmetries and ψ_k and Ψ_k are the generating functions of conservation laws for the potential Korteweg–de Vries equation

$$s_{t_1}=\phi_1$$

and the Korteweg–de Vries equation

$$T_{t_1}=\Phi_1,$$

respectively. We conclude that the opposite sides of diagram (131) are identified with the one-step vertical shift.

Corollary 46. We conclude from diagram (131) that the symmetries Φ_k of the bi-Hamiltonian hierarchy for the Korteweg–de Vries equation are the gradients ψ_k of the Hamiltonians $h_k[s]$ for the potential Korteweg–de Vries equation (92), and *vice versa*.

The equations $\mathcal{E}_{\text{KdV}}$ and $\mathcal{E}_{\text{pKdV}}$ share the same set of the Hamiltonians with the densities $h_k[T]$ and $h_k[s]$, respectively. Thence we obtain the following property of symmetry subalgebra $\mathfrak{A}$ of the Toda equation:

Theorem 47. *The generators φ_k of the commutative Lie algebra $\mathfrak{A}$ are the Noether symmetries of the Toda equation*:

$$\varphi_k \in \operatorname{sym} \mathcal{L}_{\text{Toda}}.$$

Proof. Indeed, we have

$$\mathfrak{A} \ni \varphi_k = \square(\phi_{k-1}) = \square \circ \mathbf{E}_T(h_k[T]) \in \operatorname{sym} \mathcal{L}_{\text{Toda}} \subset \operatorname{sym} \mathcal{E}_{\text{Toda}}$$

by Remark 6 on page 42. $\square$

Remark 15. The conservation laws $[\eta_k]$ associated with the generating sections

$$\psi_{k+1}^{\text{Toda}} = \hat{\kappa} \cdot \square \circ \mathbf{E}_T(h_k)$$

(see Eq. (69) on page 42 and diagram (126)) are exactly the Hamiltonians $h_k \, \mathrm{d}x$ of the higher Korteweg–de Vries equations $s_{t_k} = \phi_k$.

This fact, $[h_k \, \mathrm{d}x] \in \bar{H}^1(\mathcal{E}_{\text{Toda}})$, was noted in a similar situation in [22, §10] but required a nontrivial proof there.

By Lemma 44, the densities h_k of the Hamiltonians for both Korteweg–de Vries equations (91) and (92) are conserved on the corresponding higher analogs $T_{t_k} = D_x(\phi_k)$ and $s_{t_k} = \phi_k$ of these equations. We have

$$\bar{D}_{t_k}(h_k) = \bar{D}_x(\Omega_k^{\text{KdV}}).$$

We claim that the hierarchy $\mathfrak{B}$ is composed by conserved densities for the hierarchy $\mathfrak{A}$ of potential modified equation (93). We need the following useful lemma to prove this claim.

Lemma 48 ([63])**.** *The relation*

$$\mathbf{E}(\ni_\varphi(\mathcal{L})) = \ni_\varphi(\mathbf{E}(\mathcal{L})) + \ell_\varphi^*(\mathbf{E}(\mathcal{L}))$$

holds for any $\varphi \in \varkappa$ and $\mathcal{L} \in \bar{\Lambda}^n(\pi)$.

Proof. Suppose $\Delta \in \mathcal{C}\mathrm{Diff}(\varkappa, \bar{\Lambda}^n(\pi))$. By multiple integrating by parts, we transform the expression $\ell_{\Delta(\varphi)}$ to

$$\ell_{\Delta_0(\varphi)} + D_x \circ \Delta'(\varphi),$$

where the order of Δ_0 is zero and $\Delta'(\varphi) \in \mathcal{C}\mathrm{Diff}(\varkappa, \bar{\Lambda}^n(\pi))$; then by using Eq. (14) we obtain

$$\mathbf{E}(\Delta(\varphi)) = \ell_\varphi^*(\Delta^*(1)) + \ell_{\Delta^*(1)}^*(\varphi)$$

for any section $\varphi \in \varkappa$. Now, let $\mathcal{L} \in \bar{\Lambda}^n(\pi)$ be a form and put $\Delta = \ell_{\mathcal{L}} \colon \varkappa \to \bar{\Lambda}^n(\pi)$. The linearization $\ell_{\mathbf{E}(\mathcal{L})} = \ell_{\mathbf{E}(\mathcal{L})}^*$ of the image of the Euler operator is self–adjoint, hence we obtain the equality

$$\mathbf{E}(\ell_{\mathcal{L}}(\varphi)) = \ell_\varphi^*(\mathbf{E}(\mathcal{L})) + \ell_{\mathbf{E}(\mathcal{L})}(\varphi),$$

whence follows Lemma 48. $\square$

Remark 16. In particular, from this lemma it follows that any Noether symmetry $\varphi_{\mathcal{L}}$ of a Lagrangian $\mathcal{L}$ (we recall that the condition $\partial_{\varphi}(\mathcal{L}) = 0$ holds on $J^\infty(\pi)$ in this case) is also a symmetry of the Euler equation (15) that is assigned to $\mathcal{L}$, *i.e.*,

$$\operatorname{sym}\mathcal{L} \subseteq \operatorname{sym}\mathcal{E}.$$

Lemma 48 gives an explanation why the inverse statement is wrong.

Proposition 49 ([51]). *Suppose $k \geq 0$ is arbitrary. Then the kth term $\phi_k = \mathbf{E}_T(h_k\,\mathrm{d}x)$ of the hierarchy $\mathfrak{B}$ is a conserved density for the kth higher potential modified Korteweg–de Vries equation.*

Proof. First, we recall Theorem 35 that describes the correlation of the times within the hierarchies $\mathfrak{A}$ and $\mathfrak{B}$. Second, we apply Lemma 48 that provides the rule to commutate the Euler operator and an evolutionary derivation. Third, we recall that the higher symmetries ϕ_k are independent of the jet variable s. Thence we obtain

$$\begin{aligned}\bar{D}_{t_k}(\phi_k) = \partial_{\phi_k}(\mathbf{E}_T(h_k)) = \underbrace{\mathbf{E}_T(\overbrace{\partial_{\phi_k}(h_k)}^{\bar{D}_{t_k}(h_k)=\bar{D}_x(\Omega_k^{\mathrm{KdV}})})}_{\equiv 0} - \ell^*_{\phi_k}(\mathbf{E}_T(h_k)) = \\ = -\sum_{j>0}(-1)^j\,\bar{D}_x^j \circ \left(\frac{\partial \phi_k}{\partial s_j}\cdot \mathbf{E}_T(h_k)\right) \in \operatorname{im}\bar{D}_x.\end{aligned}$$

Therefore the density ϕ_k is conserved on the equation $\mathcal{E}_{(k)} = \{u_{t_k} = \varphi_k\}$. □

Remark 17. The second Hamiltonian structure

$$\hat{B}_2 = -\beta\,\bar{D}_x^3 + \bar{D}_x \circ T + T\cdot\bar{D}_x,$$

for the equation $\mathcal{E}_{\mathrm{KdV}}$ endowes the Fourier coefficients $\mathfrak{t}_k$ of the energy–momentum component (see Eq. (58))

$$T = \sum_{k\in\mathbb{Z}} \frac{\mathfrak{t}_k}{x^{k+2}}$$

for the Toda equation (55) with the structure of the Virasoro algebra ([39]) with the central charge $c = -\frac{3}{4}\beta$. We have

$$\begin{aligned}2\pi\boldsymbol{i}\,[\mathfrak{t}_n,\mathfrak{t}_m] &= 2(n-m)\,\mathfrak{t}_{n+m} - \beta\cdot(n^3-n)\,\delta_{n+m,0},\\ [\mathfrak{t}_k,\beta] &= 0.\end{aligned}\tag{132}$$

We emphasize that the known property [8, 78] of the integral T is adapted to the case of the Toda equation (55) associated with a nondegenerate symmetrizable matrix K that is not necessarily the Cartan matrix of a semisimple Lie algebra. The central charge c is expressed by using the constant $\beta = \sum_i a_i\Delta^i$ that depends on the symmetrizator $\vec{a}$. See Remark 5 on page 38 for a similar situation.

In the paper [52], we considered a class of generalizations of the Virasoro algebra, see Eq. (132), such that the determining relations for those generalizations are given by an N-ary skew–symmetric bracket and $N \geq 2$.

7.2. **The analogs of the modified Korteweg–de Vries equation.** In this subsection, we investigate the relation between the potential and nonpotential modified Korteweg–de Vries equations that correspond to the symmetry subalgebra $\mathfrak{A} \subset \operatorname{sym} \mathcal{L}_{\text{Toda}} \subset \operatorname{sym} \mathcal{E}_{\text{Toda}}$ of the Euler type Toda equations (55). We analyse the properties of the Miura transformation $T = T(\vartheta, \vartheta_x)$ that maps solutions of the modified Korteweg–de Vries equation (127) to solutions of Eq. (91) and thence we establish the invariant nature of the operator $\square$, see Eq. (62) on page 38, that appeared in description of the structure of symmetries for the Toda equations. Also, we construct the Hamiltonian structures for the hierarchy $\mathfrak{A}$ and prove that modified equations (93) and (127) share the Hamiltonians $[h_k \, dx]$ with the Korteweg–de Vries equations (91) and (92).

We recall that the dependent variable ϑ introduced in Eq. (128) satisfies the modified Korteweg–de Vries equation (127) and its higher analogs $\mathcal{E}_{\text{mKdV}(k)}$. By Corollary 38 on page 59, all right-hand sides in the equations $\mathcal{E}_{\text{mKdV}(k)}$ are local with respect to ϑ.

Similarly to Sec. 7.1, we establish identifications between the higher symmetries of the potential equation $\mathcal{E}_{\text{pmKdV}}$ and the generating sections of conservation laws for the equation $\mathcal{E}_{\text{mKdV}}$, and *vice versa*:

$$
\begin{tikzcd}
 & & & \cdots & \cdots \\
\mathcal{H}_k[\boldsymbol{u}] \arrow[r, "\mathbf{E}_{\boldsymbol{u}}"] \arrow[drr, "-\,\mathrm{id}"'] & \underline{\underline{\vec{\psi}_k}} \arrow[urr, "A_2"] \arrow[drr, "A_1=\hat{\kappa}^{-1}\circ D_x^{-1}"] & & & \vartheta_{t_{k+1}} = \varphi_{k+1}^{\text{mKdV}} = \vec{\psi}_{k+1} \arrow[u, "R_{\text{mKdV}}"] \\
 & & \mathcal{H}_k[\vartheta] \arrow[r, "\mathbf{E}_\vartheta"] & \boldsymbol{u}_{t_k} = \vec{\varphi}_k = \vec{\psi}_k^{\text{mKdV}} \arrow[uu, "R_{\text{pmKdV}}"] \arrow[ur, "\hat{A}_2"] \arrow[dr, "\hat{A}_1=A_1^{-1}"] & \\
\mathcal{H}_{k-1}[\boldsymbol{u}] \arrow[r, "\mathbf{E}_{\boldsymbol{u}}"] \arrow[drr, "-\,\mathrm{id}"'] & \underline{\vec{\psi}_{k-1}} \arrow[urr, "A_2"] \arrow[drr, "A_1"] & & & \underline{\underline{\vartheta_{t_k} = \varphi_k^{\text{mKdV}} = \vec{\psi}_k}} \arrow[uu, "R_{\text{mKdV}}=R^*_{\text{pmKdV}}"'] \\
 & & \mathcal{H}_{k-1}[\vartheta] \arrow[r, "\mathbf{E}_\vartheta"] & \boldsymbol{u}_{t_{k-1}} = \vec{\varphi}_{k-1} = \vec{\psi}_{k-1}^{\text{mKdV}} \arrow[uu, "R_{\text{pmKdV}}"'] \arrow[ur, "\hat{A}_2"] \arrow[dr, "\hat{A}_1"] & \\
 & & & & \underline{\vartheta_{t_{k-1}} = \varphi_{k-1}^{\text{mKdV}} = \vec{\psi}_{k-1}.} \arrow[uu, "R_{\text{mKdV}}"'] \\
 & & & \cdots \arrow[uu, "R_{\text{pmKdV}}=R^*_{\text{mKdV}}"] \arrow[ur, "\hat{A}_2"] & \cdots \arrow[u, "R_{\text{mKdV}}"']
\end{tikzcd}
$$

The Lie algebras $\mathfrak{A}$ and $\mathfrak{B}$ are commutative, see Lemma 39 and Theorem 41. Therefore, Jacobi's identities (7b) are valid for the operators $A_{1,2}$ and $B_{1,2}$ that are either induced by relations (121) for the subscript 1 or obtained from the recursions (123) by using the Magri scheme (125) for the subscript 2. By construction, the recursion operators $R_{\mathrm{p(m)KdV}}$ admit the standard decompositions

$$R_{\mathrm{pmKdV}} = A_2 \circ A_1^{-1} = \square \circ D_x^{-1} \circ \ell_T,$$
$$R_{\mathrm{pKdV}} = B_2 \circ B_1^{-1},$$

where the skew–symmetric factors $A_{1,2}$ and $B_{1,2}$ are

$$\begin{cases} A_1 = \hat{\kappa}^{-1} \circ D_x^{-1}, \\ A_2 = \square \circ D_x^{-1} \circ \square^*, \end{cases} \qquad \begin{cases} B_1 = D_x^{-1}, \\ B_2 = -\beta\, D_x + s_1 \cdot D_x^{-1} + D_x^{-1} \circ s_1. \end{cases}$$

We denote by ℓ_T^u the linearization of functional (58) with respect to the dependent variables u and by ℓ_T^ϑ its linearization with respect to ϑ. The operator ℓ_T^u is given in Eq. (74) on page 46, and we have

$$\ell_T^\vartheta = \Bigl(\ldots, \sum_\sigma \frac{\partial T}{\partial \vartheta^i_\sigma}\, D_\sigma, \ldots\Bigr).$$

Lemma 50. *The relation $\square^* = \ell_T^\vartheta$ holds. Moreover,*

$$\ell_T^{\boldsymbol{u}} = \square^* \circ \ell_\vartheta^{\boldsymbol{u}}.$$

Proof. The verification of the first statement is straightforward. Expressing u_x via ϑ, $u_x = \kappa^{-1}\vartheta$, we write down functional (58) in terms of ϑ: we have

$$T = \tfrac{1}{2} \sum_{l,m} \kappa^{lm} \vartheta^l \vartheta^m - \sum\nolimits_l \Delta^l \cdot \vartheta^l_x,$$

where $\kappa^{-1} = \|\kappa^{lm}\|$ and the identity

$$\sum_{i=1}^r a_i \cdot \kappa^{ij} = \Delta^j$$

is used. Consequently, we obtain

$$\square^* = {}^t\bigl(\kappa^{-1} \cdot \vartheta - \vec{\Delta} \cdot \bar{D}_x\bigr).$$

The second statement follows from the definition of the linearization on page 8. $\square$

Several decompositions follow from this lemma. For example, we obtain

$$\ell_T = \square^* \circ A_1^{-1} = \square^* \circ \hat{A}_1 = \square^* \circ D_x \circ \hat{\kappa} = \square^* \circ \ell_\vartheta^{\boldsymbol{u}}.$$

The recursion operator R_{pKdV} is factorized to the product

$$R_{\mathrm{pKdV}}(\phi_{k-1}) = \mathfrak{I}_{\square(\phi_{k-1})}(s) = \ell_s(\square(\phi_{k-1})) =$$
$$= D_x^{-1} \circ \ell_T \circ \square(\phi_{k-1}) = B_1 \circ \square^* \circ \hat{A}_1 \circ \square(\phi_{k-1})$$

of the Hamiltonian operators glued together by the operators $\square$ and $\square^*$.

Proposition 51 ([51, 60]). *Each Noether symmetry*

$$\varphi_{\mathcal{L}} = \square \circ \mathbf{E}_T(Q(x, \mathbf{T})) \in \operatorname{sym} \mathcal{L}_{\mathrm{Toda}}$$

of the Toda equation associated with a nondegenerate symmetrizable matrix K is Hamiltonian with respect to the Hamiltonian structure $A_1 = \kappa^{-1} \cdot D_x^{-1}$ and the Hamiltonian $\mathcal{H} = [Q(x, \mathbf{T})]$:

$$\varphi_{\mathcal{L}} = A_1 \circ \mathbf{E}_u(\mathcal{H}).$$

The proof of Proposition 51 follows from the definition of the Euler operator, $\mathbf{E}(H\,\mathrm{d}x) = \ell_H^*(1)$, and Lemma 50.

Remark 18. The propetry of the Noether point symmetries

$$\varphi_0^f = \square \circ \mathbf{E}_T(T \cdot f(x)\,\mathrm{d}x)$$

of the Toda equation to be Hamiltonian,

$$\varphi_0^f = A_1 \circ \mathbf{E}_u(T \cdot f(x)\,\mathrm{d}x),$$

was established in [77]. One easily verifies that equation (93) is also Hamiltonian with respect to the cohomology class h_0, see diagram (131) on page 74:

$$\varphi_1 = A_1 \circ \mathbf{E}_u(h_0\,\mathrm{d}x).$$

Now we extend Remark 18 onto the whole hierarchy $\mathfrak{A}$. Here we formulate the most remarkable relation between the hierarchy $\mathfrak{A}$ for the potential modified Korteweg–de Vries equation (93) and the hierarchy $\mathfrak{B}$ for scalar equation (91).

Theorem 52 ([51, 60]). *The Hamiltonian $[h_k\,\mathrm{d}x]$ for the* kth *higher Korteweg–de Vries equation is the Hamiltonian for the Noether symmetry $\varphi_k \in \mathfrak{A}$ for any integer $k \geq 0$.*

This theorem follows from Proposition 51 and Remark 15 on page 75 that assign the Noether symmetries $\varphi_k \in \mathfrak{A}$ to the conserved densities h_k.

Remark 19. The Lagrangian representation of the Hamiltonian equations $\mathcal{E}_{(k)}$ within the hierarchy $\mathfrak{A}$ is

$$\mathcal{E}_{(k)} = \Big\{ A_1 \circ \mathbf{E}_u\big(\big[-\tfrac{1}{2} \sum_{i,j} \kappa_{ij} u_x^i u_{t_k}^j - h_{k-1} \big]\,\mathrm{d}x \big) = 0 \Big\}, \quad k \geq 0.$$

Here we complete the description of the Magri scheme for the equation (93). The Toda equation (118) is the first nonlocal term of the symmetry tower below:

$$\begin{tikzcd}
 & & \cdots & \cdots & & \\
 & \cdots & & & \cdots & \\
 & & \boldsymbol{u}_{t_{k+1}} = \vec{\varphi}_{k+1} \arrow[u, "R_{\mathrm{pmKdV}}"] \arrow[uur, "\ell_s"'] & s_{t_k} = \phi_k \arrow[l, "\square"'] \arrow[u, "R_{\mathrm{pKdV}}"'] & & \\
h_k \arrow[r, "\mathbf{E}_{\boldsymbol{u}}"] & \vec{\psi}_k \arrow[uu, "R^*_{\mathrm{pmKdV}}"] \arrow[urr, "A_2"] \arrow[drr, "A_1"] & & & \psi_{k-1} \arrow[uu, "R^*_{\mathrm{pKdV}}"'] \arrow[ul, "B_2"'] \arrow[dl, "B_1"] & h_{k-1} \arrow[l, "\mathbf{E}_s"'] \\
 & & \boldsymbol{u}_{t_k} = \vec{\varphi}_k \arrow[uu, "R_{\mathrm{pmKdV}}"] \arrow[uur, "\ell_s"'] & s_{t_{k-1}} = \phi_{k-1} \arrow[l, "\square"'] \arrow[uu, "R_{\mathrm{pKdV}}"'] & & \\
h_{k-1} \arrow[r, "\mathbf{E}_{\boldsymbol{u}}"] & \vec{\psi}_{k-1} \arrow[uu, "R^*_{\mathrm{pmKdV}}"] \arrow[ur, "A_2"] \arrow[dr, "A_1"] & & & \psi_{k-2} \arrow[uu, "R^*_{\mathrm{pKdV}}"'] \arrow[ul, "B_2"'] \arrow[dl, "B_1"] & h_{k-2}. \arrow[l, "\mathbf{E}_s"'] \\
 & & \boldsymbol{u}_{t_{k-1}} = \vec{\varphi}_{k+1} \arrow[uu, "R_{\mathrm{pmKdV}}"] \arrow[uur, "\ell_s"'] & s_{t_{k-2}} = \phi_{k-2} \arrow[l, "\square"'] \arrow[uu, "R_{\mathrm{pKdV}}"'] & &
\end{tikzcd}$$

Finally, we have the following assertion

Proposition 53 ([51, 60])**.** (1) *The substitution*

$$T = T(\vartheta, \vartheta_x)\colon \mathcal{E}_{\mathrm{mKdV}} \xrightarrow{T(\vartheta,\vartheta_x)} \mathcal{E}_{\mathrm{KdV}} \tag{133}$$

is the Miura transformation between the higher equations

$$\mathcal{E}_{\mathrm{mKdV}(k)} = \{\vartheta_{t_k} = D_x \cdot \hat{\kappa}(\varphi_k)\}$$

and

$$\mathcal{E}_{\mathrm{KdV}(k)} = \{T_{t_k} = D_x(\phi_k)\}.$$

(2) *The operator*

$$\square^* = \ell_T^{\vartheta}\colon \varphi \mapsto \mathfrak{I}_{\varphi}(T)$$

maps the Lie algebra $\operatorname{sym}\mathcal{E}_{\mathrm{mKdV}} \ni \varphi$ *of symmetries of the equation* $\mathcal{E}_{\mathrm{mKdV}}$ *into the symmetry Lie algebra* $\operatorname{sym}\mathcal{E}_{\mathrm{KdV}}$ *for the Korteweg–de Vries equation* (91)*, see Example* 15*.*

(3) *The operator* $\square = \square^{**}$ *maps the generating sections of conservation laws which are dual to the symmetries,*

$$\square\colon \phi_k = \mathbf{E}_T(h_k\,\mathrm{d}x) \mapsto \varphi_k \in \mathfrak{A},$$

in the opposite direction.

Remark 20. From diagrams (104) and (131) we see that the successive application of the mappings $\square^* = \ell_T^\vartheta$ and $\square = \square^{**}$ does not preserve the index of a higher symmetry:

$$\begin{tikzcd} \vec{\psi}_k = \varphi_k^{\text{mKdV}} \arrow[r, "\square^*"] & \psi_k = \Phi_k \arrow[r, "B_1"] & \phi_k \\ \vec{\varphi}_k = \vec{\psi}_k^{\text{mKdV}} \arrow[u, "\hat{A}_1"] & & \phi_{k-1} \arrow[ll, "\square=\square^{**}"'] \arrow[u, "R_{\text{pKdV}}"'] \end{tikzcd} \qquad (134)$$

and the recursion operator R_{pKdV} measures the difference in their action.

Part II. Group–theoretic properties of the mathematical physics equations: methods and applications

Within Part II of the present paper, we illustrate the cohomological concepts and algorithms in the geometry of differential equations ([10, 62, 63, 94]). Namely, we apply these methods to the study of the properties of the dispersionless Toda equation, the multi–component nonlinear Schrödinger equation, the Liouville equation, and related systems. Invariant solutions, Noether's symmetries, local and nonlocal conservation laws, weakly nonlocal recursion operators, parametric families of Bäcklund (auto)transformations, and zero–curvature representations are the structures, which are obtained for these equations of mathematical physics.

Chapter 3. Symmetries, solutions, and conservation laws for differential equations

In this chapter, we consider two examples of applications of the PDE geometry methods; namely, we analyse the geometric properties of the dispersionless Toda equation ([50, 56]) and obtain the structures ([55]) for the multi-component analog of the nonlinear Schrödinger equation, which is related with the former one (see [16]).

8. Nonlinear Schrödinger equation

In this section, we study the properties of the m-component analogs of the nonlinear Schrödinger equation ([98, 1, 92])

$$\Psi_t = \boldsymbol{i}\Psi_{xx} + \boldsymbol{i}f(|\Psi|)\,\Psi, \tag{135}$$

where Ψ is an m-component vector ($m \geq 1$), $\boldsymbol{i} = \sqrt{-1}$, and $f \in C^1(\mathbb{R})$ is a smooth function.

The scalar ($m = 1$) nonlinear Schrödinger equation describes the propagation of light pulses or wave packets in the linear dissipation and nonlinear self-focusing media, *e.g.*, in optical fibers, nonlinear cristals and gases, multicomponent Bose–Einstein condensates at zero temperature, *etc*. The collective evolution of a complex of space–incoherent solitons in a nonlinear media with a Kerr–like nonlinearity is described by system (135) of the nonlinear Schrödinger equations, where $\Psi = {}^t(\Psi^1, \ldots, \Psi^m)$ are the amplitudes of the light wave,

$$\mathcal{I} = \sum_{i=1}^{m} |\Psi^i|^2$$

is the density of energy, t is the coordinate along the direction of the wave propagation, and x is the coordinate along the front of the wave.

Suppose $f = \mathrm{id}$, then this equation is known ([98]) to admit a commutative bi-Hamiltonian hierarchy of higher symmetries and an infinite number of conserved densities in involution. Still, for an arbitrary f

this is not so. In the sequel, we obtain the symmetry algebra of the latter equation in the physically actual case of a homogeneous function f of weight Δ; also, we desribe the set of m^2 conserved currents, which generalize the previously known m separate conservation laws of energy of an ith mode and the conservation law of total momentum of the system. Those conservation laws correspond to two types of Hamiltonian symmetries of the Schrödinger equation, the scaling transformation and the translation.

So, consider the m-component analog of the nonlinear Schrödinger equation, the multi–soliton complex ([1])

$$\begin{aligned} F^k &\equiv \Psi^k_t - \boldsymbol{i}\Psi^k_{xx} - \boldsymbol{i}f(\mathcal{I})\cdot\Psi^k = 0, \\ \bar{F}^k &\equiv \bar{\Psi}^k_t + \boldsymbol{i}\bar{\Psi}^k_{xx} + \boldsymbol{i}f(\mathcal{I})\cdot\bar{\Psi}^k = 0, \quad 1\le k\le m,\ \mathcal{I}\equiv\Psi\bar{\Psi}. \end{aligned} \tag{136}$$

This equation is Hamiltonian for an arbitrary function f:

$$\begin{pmatrix}\Psi\\ \bar{\Psi}\end{pmatrix}_t = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\delta\mathcal{H}_{\text{NLS}}/\delta\Psi\\ \delta\mathcal{H}_{\text{NLS}}/\delta\bar{\Psi}\end{pmatrix}, \tag{137}$$

where the density H_{NLS} of the Hamiltonian $\mathcal{H}_{\text{NLS}} = [H_{\text{NLS}}\,\mathrm{d}x]$ is

$$H_{\text{NLS}} = -\boldsymbol{i}\Psi_x\bar{\Psi}_x + \boldsymbol{i}\int^{\mathcal{I}} f(\mathcal{I})\,\mathrm{d}\mathcal{I}.$$

We also note that H_{NLS} is a conserved density for Eq. (136), quite similarly to Eq. (11) for the dispersionless Toda equation or to Eq. (119) on page 68 for the Toda equation (55). From the Hamiltonian representation (137) we deduce that the complex conjugate variables $\bar{\Psi}$ are the canonically conjugate variables (the momenta) for the dynamical coordinates Ψ; in Sec. 5 we investigated a generalization of this situation.

Equation (135) is known to possess the following property (see [1] and references therein): in addition to the conservation of the total energy $\Psi\bar{\Psi}$, there are m separately conserved densities

$$Q_i = \Psi^i\cdot\bar{\Psi}^i, \qquad \bar{D}_t(Q_i) = 0. \tag{138}$$

Physically speaking, there is no energy transfer between the modes Ψ^i. Still, this observation about the properties of the nonlinear Schrödinger equation is uncomplete, since these m integrals of motion (138) are only particular instances within the set of m^2 currents ([55])

$$\eta_{ij} = \Psi^i\bar{\Psi}^j\,\mathrm{d}x + \boldsymbol{i}\left(\Psi^i_x\,\bar{\Psi}^j - \Psi^i\,\bar{\Psi}^j_x\right)\,\mathrm{d}t, \qquad \bar{\mathrm{d}}_h\eta_{ij} = 0. \tag{139}$$

These currents are conserved on Eq. (135) with an arbitrary nonlinearity $f(\mathcal{I})$. They reflect the correlation between the intensities of the light modes with *different* indexes i and j, $1\le i,\ j\le m$. The set of conservation laws (139) remained unnoticed in the papers [1, 92], and the arising conservation restrictions seem to have not been taken into consideration by the authors when making computer experiments.

The generating sections of the conservation laws $[\eta_{ij}]$ defined in Eq. (139) are

$$\vec{\psi}_{(ij)} = {}^t(\psi_{(ij)}, \bar{\psi}_{(ij)}),$$

where

$$\psi^i_{(ij)} = \bar{\Psi}^j, \quad \bar{\psi}^j_{(ij)} = \Psi^i,$$

and

$$\psi^{i'}_{(ij)} = \bar{\psi}^{j'}_{(ij)} = 0 \text{ for } i' \neq i,\ j' \neq j.$$

The canonical Hamiltonian structure

$$\Gamma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

assigns the point symmetries

$$\vec{\varphi}_{(ij)} = {}^t\bigl(\varphi_{(ij)}, \bar{\varphi}_{(ij)}\bigr)\colon\ \varphi^i_{(ij)} = \Psi^j,\ \bar{\varphi}^j_{(ij)} = -\bar{\Psi}^i$$

to these sections. Suppose $i = j$, then we get the gauge symmetry ([98])

$$A_\lambda(\vec{\varphi}_{(ii)})\colon \Psi^i \mapsto \exp(\lambda)\Psi^i,\ \bar{\Psi}^i \mapsto \exp(-\lambda)\bar{\Psi}^i. \tag{140}$$

The Hamiltonians of the symmetries $\vec{\varphi}_{(ii)}$ are the conserved densities Q_i in Eq. (138).

In the paper [98], the case $f(\mathcal{I}) = \mathcal{I}$ of cubic nonlinearity in Eq. (135) was analysed. Then, the cubic m-component equation (135) admits the recursion operator

$$R_{\text{NLS}} = \begin{pmatrix} -D_x & 0 \\ 0 & D_x \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -\Psi \\ \bar{\Psi} \end{pmatrix} \cdot D_x^{-1} \circ \bigl(\bar{\Psi}, \Psi\bigr) - \frac{1}{2}\sum_{i,j} \vec{\varphi}_{(ij)} \cdot D_x^{-1} \circ \vec{\psi}_{(ij)}.$$

This operator generates an infinite sequence of the commuting local higher symmetries of Eq. (135), starting with scaling symmetry (140):

$$\begin{tikzcd}
Q_i\,\mathrm{d}x \arrow[d, "\mathbf{E}_{\vec{\Psi}}"] & & \mathcal{H}_{\text{NLS}} \arrow[d, "\mathbf{E}_{\vec{\Psi}}"] & & \mathcal{H}_4 \arrow[d, "\mathbf{E}_{\vec{\Psi}}"] & \\
\vec{\psi}_{(ii)} \arrow[r, "R^*_{\text{NLS}}"] \arrow[d, "\Gamma_1"] & \vec{\psi}_1 \arrow[r, "R^*_{\text{NLS}}"] \arrow[d, "\Gamma_1"] & \vec{\psi}_2 \arrow[r, "R^*_{\text{NLS}}"] \arrow[d, "\Gamma_1"] & \vec{\psi}_3 \arrow[r, "R^*_{\text{NLS}}"] \arrow[d, "\Gamma_1"] & \vec{\psi}_4 \arrow[r, "R^*_{\text{NLS}}"] \arrow[d, "\Gamma_1"] & \ldots \\
\vec{\varphi}_{(ii)} \arrow[r, "R_{\text{NLS}}"] & \vec{\Psi}_x \arrow[r, "R_{\text{NLS}}"] & \vec{\Psi}_t \arrow[r, "R_{\text{NLS}}"] & \vec{\varphi}_3 \arrow[r, "R_{\text{NLS}}"] & \vec{\varphi}_4 \arrow[r, "R_{\text{NLS}}"] & \ldots.
\end{tikzcd} \tag{141}$$

This diagram is an example of a sequence of Hamiltonian symmetries such that there are no Hamiltonians for a half of these symmetries, $\vec{\varphi}_{2k+1}$: we have $\vec{\psi}_{2k+1} \notin \operatorname{im} \mathbf{E}_{\vec{\Psi}}$.

Remark 21. We have obtained the formulae for the generating sections $\vec{\psi}_{(ij)}$ of conservation laws (139) and the Hamiltonian symmetries $\vec{\varphi}_{(ij)}$. Now we conclude that the recursion operator R_{NLS} is *weakly nonlocal* ([61]), *i.e.*, this operator is decomposable to the form

$$R = \text{local part} + \sum\nolimits_\alpha \varphi_\alpha \circ D_x^{-1} \circ \psi_\alpha, \tag{142}$$

where φ_α are symmetries and ψ_α are generating sections of conservation laws. We recall that in the preceding chapter we used the recursion operators' decompositions similar to (142), and then we generated the commuting symmetry sequences for the Toda equations and established nontrivial properties of these sequences.

We finally note that the symmetries $\vec{\varphi}$ that are independent on x and t (only such symmetries were considered in [98]) do not exhaust the whole algebra of classical symmetries for the nonlinear Schrödinger equation (135). Moreover, the resulting symmetry algebra of this equation is noncommutative.

Example 21. Suppose there are no restrictions for the nonlinearity function $f(\mathcal{I})$ in Eq. (136). Then the point symmetry algebra of the latter equation is generated by the fields whose sections are

$$\vec{\varphi}_{(ij)},\ \ \vec{\Psi}_x,\ \ \vec{\Psi}_t,\ \begin{pmatrix}\varphi^i\\ \bar{\varphi}^i\end{pmatrix} = \begin{pmatrix}2t\,\Psi^i_x - \boldsymbol{i}x\,\Psi^i\\ 2t\,\bar{\Psi}^i_x + \boldsymbol{i}x\,\Psi^i\end{pmatrix}.$$

Further on, suppose additionally that f is homogeneous,

$$f(\lambda\mathcal{I}) = \lambda^\Delta \cdot f(\mathcal{I}).$$

Then equation (135) acquires the scaling symmetry

$$\begin{pmatrix}\varphi^i\\ \bar{\varphi}^i\end{pmatrix} = \begin{pmatrix}2\Psi^i + \Delta\,x\,\Psi^i_x + 2\Delta\,t\,\Psi^i_t\\ 2\bar{\Psi}^i + \Delta\,x\,\bar{\Psi}^i_x + 2\Delta\,t\,\bar{\Psi}^i_t\end{pmatrix}.$$

9. The dispersionless Toda equation

The dispersionless Toda equation is an analog of Eq. (55) with the continuous variation of the index j that enumerates the dependent variables u^j. In this section, the classical symmetry algebra of the dispersionless Toda equation is computed and 5 classes of conservation laws for the latter equation are reconstructed; also, some questions of the Lagrangian formalism with higher-order derivatives are discussed.

Consider the hyperbolic Toda equations

$$\boldsymbol{u}_{xy} = \exp(K\boldsymbol{u}), \qquad \boldsymbol{u}'_{xy} = K \cdot \exp(\boldsymbol{u}'), \tag{143}$$

associated with the type A_{r-1} Lie algebras with the Cartan matrices K, introduce an additional independent variable $z \in \mathbb{R}$, and extend the values of the discrete index $j \in [1, r]$, which enumerates the dependent

variables u^j, onto the whole line $\mathbb{R}$. Also, let the rank r tend to infinity: $r \to \infty$. Then for any section $\boldsymbol{u}$ of the bundle π we set

$$u^j = \boldsymbol{u}(x, y, z)\big|_{z=j\varepsilon},$$

where ε is the cell step.

As $r \to \infty$ and $\varepsilon \to +0$, the continuous limit equation, which is refered as either the dispersionless Toda equation, or the heavenly equation, or the Boyer–Finley equation ([11]), also appeares in many models, *e.g.*, in gravity ([87, 11]) as a reduction of the antiselfdual vacuum Einstein equation (ASDVEE). The heavenly equation $\mathcal{E}_{\text{heav}}$ exists due to the special structure (see Eq. (45)) of the Cartan matrices $K = \|k_{ij}\|$ of the type A_r algebras. As a matter of fact, instead of the Cartan $(r \times r)$-matrices K one should consider ([87, 88]) the Cartan operator $\hat{K} = -D_z^2$. Then the scalar equations, which originate from Eq. (143), are ([2, 11])

$$\hat{\mathcal{E}} = \{\hat{F} \equiv u_{xy} - \exp(-u_{zz}) = 0\} \tag{144a}$$

$$u'_{xy} = -D_z^2 \circ \exp(u'); \tag{144b}$$

the Cartan operator $\hat{K}$ defines the sign "$-$" in the one–dimensional equations

$$u_{\tau\tau} = \exp(-u_{zz}), \qquad u'_{\tau\tau} = -D_z^2 \circ \exp(u') \tag{145}$$

that appeared in the Introduction.

Following the papers [87, 16], we consider the limit

$$\lim_{\varepsilon \to +0} \lim_{r \to \infty} L_{\text{Toda}} = D_z^{-1}\left(\tfrac{1}{2} u_x u_{yzz} - \exp(-u_{zz})\right)$$

of the Lagrangian density $\mathcal{L}_{\text{Toda}}$, defined in Eq. (56), as $r \to \infty$ and $\varepsilon \to +0$. Then the Lagrangian $\mathcal{L}_{\text{Toda}}$ itself is mapped to the functional

$$\hat{\mathcal{L}} = \iint \mathrm{d}x \mathrm{d}y \int \text{Ł}\, \mathrm{d}z$$

with the density

$$\text{Ł} = -\tfrac{1}{2}\, u_{xz} u_{yz} - \exp(-u_{zz})$$

that depends on the second derivatives of the sections $u = u(x, y, z)$. Applying the Euler operator $\mathbf{E}_u$ to $\hat{\mathcal{L}}$, we obtain the equation

$$\mathcal{E}_{\text{heav}} = \left\{F_{\text{heav}} \equiv u_{xyzz} - D_z^2 \circ \exp(-u_{zz}) = 0\right\}; \tag{146}$$

we see that double integrating in z maps Eq. (146) to Eq. (144a), while the substitution $u' = -u_{zz}$ maps it to Eq. (144b).

9.1. **Symmetries and exact solutions.** Computing the symmetries $\varphi \in \operatorname{Sym} \hat{\mathcal{E}}$ of Eq. (144a) by using the analytic transformations software Jet ([72]), we get the following solution:

Proposition 54. *The point symmetries*

$$\varphi(x, y, z, u, u_x, u_y, u_z)$$

of Eq. (144a), *which solve the determining equation*

$$\bar{D}_{xy}(\varphi) + \exp(-u_{zz}) \cdot \bar{D}_z^2(\varphi) = 0,$$

are

$$\varphi_1[f] = \Big(u_x - \tfrac{1}{2}z^2\,\bar{D}_x\Big)\,f(x), \quad \bar{\varphi}_1[g] = \Big(u_y - \tfrac{1}{2}z^2\,\bar{D}_y\Big)\,g(y), \tag{147a}$$

$$\varphi_2 = -\tfrac{1}{2}zu_z + u - \tfrac{1}{2}z^2, \tag{147b}$$

$$\varphi_3 = u_z, \tag{147c}$$

$$\varphi_4[q] = q(x)\,z, \qquad \bar{\varphi}_4[\bar{q}] = \bar{q}(y)\,z, \tag{147d}$$

$$\varphi_5[r] = r(x), \qquad \bar{\varphi}_5[\bar{r}] = \bar{r}(y), \tag{147e}$$

where f, q, and r are arbitrary smooth functions of x and g, $\bar{q}$, and $\bar{r}$ are functions of y. The commutation rules for symmetries (147) *are given in the skew–symmetric table below*:

	$\varphi_1[f]$	$\bar{\varphi}_1[g]$	φ_2	φ_3	$\varphi_4[q]$	$\bar{\varphi}_4[\bar{q}]$	$\varphi_5[r]$	$\bar{\varphi}_5[\bar{r}]$
$\varphi_1[f]$	0	0	0	$\varphi_4[-f']$	$\varphi_4[-fq']$	0	$\varphi_5[-fr']$	0
$\bar{\varphi}_1[g]$		0	0	$\bar{\varphi}_4[-g']$	0	$\bar{\varphi}_4[-g\bar{q}]$	0	$\bar{\varphi}_5[-g\bar{r}']$
φ_2			0	$\varphi_3+\varphi_4[2]$	$\varphi_4[q]$	$\bar{\varphi}_4[\bar{q}]$	$\varphi_5[2r]$	$\bar{\varphi}_5[2\bar{r}]$
φ_3				0	$\varphi_5[-q]$	$\bar{\varphi}_5[-\bar{q}]$	0	0
$\varphi_4[q]$					0	0	0	0
$\bar{\varphi}_4[\bar{q}]$						0	0	0
$\varphi_5[r]$							0	0
$\bar{\varphi}_5[\bar{r}]$								0

Remark 22. 1. The operator $\hat{\Box} = u_x - \frac{1}{2}z^2\,\bar{D}_x$ in Eq. (147a) is an analog of the operator $\Box$, which was introduced in Eq. (62) on page 38.
2. The symmetries of Eq. (144b) that correspond to (147a)–(147c) were obtained in [2]; the symmetries φ_4, $\varphi_5 \in \ker D_z^2$ of the equation $\hat{\mathcal{E}}$ demonstrate the (inessential) distinction between the geometries of Eq. (144a) and Eq. (144b), this distinction is provided by the substitution $u' = -u_{zz}$.

Remark 23. In fact, equations (144) and (146) are huge inspite of an apparent compactness of the notation. Indeed, one can estimate the number of kth order internal coordinates for these equations or the total number of nontrivial relations that define the kth prolongation $\mathcal{E}^{(k)}_{\text{heav}}$. Therefore the calculation of the higher symmetry algebra $\operatorname{sym}\mathcal{E}^{\infty}_{\text{heav}}$ for Eq. (146) is really difficult.

The passage from difference–differential equations (46) associated with Cartan's matrix (45) to their *dispersionless* limit (144a) provides the additional constraint

$$u_{zzz} = 0.$$

The latter equation must be taken into consideration when describing the Lie algebra $\operatorname{sym}\mathcal{E}^\infty_{\text{heav}}$ composed by *all* symmetries of the limit dispersionless Toda equation.

9.1.1. *Constructing exact solutions to the dispersionless Toda equation.* By using the structure of the point symmetry Lie algebra $\operatorname{Sym}\mathcal{E}$ for Eq. (144a) and applying various geometric methods ([10]), the problem of constructing exact solutions to the heavenly equation was considered in [2] and [71]. In the former paper, invariant solutions of the equation $u_{xy} = \pm(\exp(u))_{zz}$ were obtained; they correspond to Eqs. (152) and (153) of the present article. A class of solutions that are non-invariant with respect to the whole Lie algebra $\operatorname{Sym}\mathcal{E}$ was pointed out in [71]. The approach of this paper is wider than in [2] in the following sence: any time an equation under consideration being reduced to an auxiliary equation for a function that depends on fewer arguments than the initial one, we construct the point symmetry Lie algebra for the new equation and obtain its invariant solutions that are parameterized by arbitrary functions. Therefore, what we get is much more than a list of particular solutions. Also, we note that the method of solving Eq. (144a) together with the constraint $\varphi_i = 0$ is similar to the scheme ([71]) used to construct *non*-invariant solutions of differential equations. Indeed, we treat the variable z in the constraints $\varphi_i = 0$ as a formal parameter in order to write down these additional equations $\varphi_i = 0$ in total differentials.

The symmetry $\varphi_1 \in \operatorname{Sym}\hat{\mathcal{E}}$. Consider the generator

$$\varphi_1 = \varphi_1[f] + \bar{\varphi}_1[g] = u_x\, f(x) - \tfrac{1}{2}z^2\, f'(x) + u_y\, g(y) - \tfrac{1}{2}z^2\, g'(y)$$

of an infinitesimal conformal symmetry of Eq. (144a); this generator depends on two arbitrary smooth functions f and g. Now we represent the invariance condition $\varphi_1 = 0$ in the characteristic form and obtain the first integral

$$t = \int^x \mathrm{d}x/f(x) - \int^y \mathrm{d}y/g(y)$$

that is independent on the variable u. In order to construct another integral C_2 we treat the base coordinate z as a parameter:

$$\tfrac{1}{2}z^2 \log f(x) + \tfrac{1}{2}z^2 \log g(y) - u = C_2(z).$$

Then, a solution $u(x, y, z)$ to the equation $\varphi_1 = 0$ is defined by the condition

$$\Pi(t, C_2(z)) = 0,$$

where Π is an arbitrary (for a while) function of two "integrals", one of which depends on z. Solving this relation with respect to u, we get

$$u = \tfrac{1}{2}z^2 \log f(x)g(y) + \Phi(t, z). \tag{148}$$

Moreover, if u is a solution to Eq. (144a), then it is necessary that the function Φ satisfies the one–dimensional equation

$$\Phi_{tt} + \exp(-\Phi_{zz}) = 0. \tag{145$'$}$$

The reduced equation is nothing else than the one–dimensional analog of Eq. (145). In a similar situation, the hyperbolic Liouville equation is reduced to solving its one–dimensional analog when one constructs solutions that are invariant with respect to the conformal symmetries of the initial Liouville equation. We emphasize that the second "integral" $C_2(z)$ of the equation $\varphi_1 = 0$ was obtained by using a nontrivial interpretation of the coordinate z as a parameter which is auxiliary with respect to the independent variables x and y. This approach seems to have not been used in [2]; in the sequel, we apply it to Eq. (145$'$) again. We use the Jet ([72]) software and compute the generators of the point symmetry Lie algebra of Eq. (145$'$); then, we obtain the invariant solutions of this equation. The result is given in

Lemma 55. *The Lie algebra of classical symmetries of Eq.* (145$'$) *is generated by eight evolutionary vector fields whose generating functions are*

$$\begin{array}{llll} \phi_1 = t\Phi_t - z^2, & \phi_2 = \Phi_t, & \phi_3 = z\Phi_z + z^2 - 2\Phi, & \phi_4 = \Phi_z, \\ \phi_5 = zt, & \phi_6 = t, & \phi_7 = z, & \phi_8 = 1. \end{array}$$

Further on, we point out the families of solutions to Eq. (145$'$) and assign the solution class (148) for the dispersionless Toda equation (see Eq. (144a)) to each of them.

The symmetry ϕ_1. Consider the equation $\phi_1 = 0$; again, we require the variable z to be a formal parameter, then we have $z^2 \log|t| - \Phi = C(z)$. Hence we express the solution Φ, substitute it into Eq. (145$'$), and solve the latter with respect to C; we finally get

$$\Phi(t,z) = z^2 \log|t| - \tfrac{1}{4} z^2 \log|z| - \tfrac{1}{8} z^2 + C_1 z + C_2,$$

where C_1 and C_2 are arbitrary constants.

The symmetry ϕ_2. One easily checks that the real ϕ_2-invariant solutions of Eq. (145$'$) do not exist at all, since the equation $\exp(-\Phi_{zz}) = 0$ is insoluble over $\mathbb{R}$. Still, we consider a more general case: namely, we seek solutions to Eq. (145$'$) that are invariant with respect to the linear combination

$$\phi_{(a:b)} = \phi_2 + (a:b)\phi_4$$

of the symmetries, where $(a:b) \in \mathbb{RP}^1$, *i.e.*, we seek the propagating wave solutions. Substituting the function $\Phi(z - (a:b)\,t)$ into Eq. (145$'$), we get the equation

$$(a:b)^2\, \Psi = -\exp(-\Psi). \tag{149}$$

Here by Ψ we denote the second derivative Φ'' of Φ with respect to its argument

$$w = z - (a:b)\,t.$$

Suppose the condition

$$-\sqrt{\mathrm{e}} < (a:b) < \sqrt{\mathrm{e}}$$

is satisfied, then the obstacle for the low–velocity $\phi_{(a:b)}$-invariant solutions to exist is clear from Eq. (149). Indeed, there is the critical (minimal) velocity $|a:b| = \sqrt{\mathrm{e}}$ and the wave solution

$$\Phi = -\tfrac{1}{2}\bigl(z \pm \sqrt{\mathrm{e}}\,t\bigr)^2 + \alpha\bigl(z \pm \sqrt{\mathrm{e}}\,t\bigr) + \beta,$$

where α, $\beta \in \mathbb{R}$. Quite remarkably, if the velocity $\sqrt{\mathrm{e}} < |a:b| < \infty$ is greater then the minimum velocity, then there are distinct roots $\Psi_{1,2}$ of Eq. (149) and Eq. (145′) admits *two* solutions

$$\Phi = \tfrac{1}{2}\Psi_{1,2}\cdot w^2 + \alpha w + \beta, \qquad w = z - (a:b)\,t, \qquad \alpha, \beta \in \mathbb{R}, \quad (150)$$

similtaneously. This pair of roots $\Psi_{1,2}$ appeares after the obvious bifurcation: no solutions of Eq. (149) and (145′) are assigned to a low inclination $(a:b)^2$ of the straight line $y = (a:b)^2\,\Psi$ on the $0\Psi y$-plane; then, at the inclination $(a:b)^2 = \mathrm{e}$ this straight line is tangent to the graph of exponent $y = -\exp(-\Psi)$ at the point $(-1, -\mathrm{e})$, while at a greater incline $\mathrm{e} < (a:b)^2 < \infty$ the point of tangency is split to the pair of distinct intersection points $(\Psi_{1,2}, (a:b)^2\Psi_{1,2})$, where $-\infty < \Psi_1 < -1 < \Psi_2 < 0$. Each admissible value of Ψ determines the second derivative of a polynomial solution (150) of Eq. (145′). A unique root $\Psi = 0$ and the ϕ_4-invariant solution of Eq. (145′) (see below) is assigned to the point $a:b = \infty$.

The symmetry ϕ_3. First, we solve the auxiliary ordinary differential equation:

$$xy'(x) - 2y = -x^2 \quad \Longrightarrow \quad y(x) = \bigl(\gamma - \tfrac{1}{2}\log x^2\bigr)\cdot x^2. \qquad (151)$$

We shall use formulae (151) twice, in order to construct the solutions of Eq. (145′) that are invariant with respect to the symmetry ϕ_3 and to get the φ_2-invariant solutions of the initial equation (144a). In these two cases, the function γ depends on two distinct sets of variables: the first case is $\gamma = \gamma(t)$ and the second case is $\gamma = \gamma(x, y)$.

Substituting the expression

$$\Phi = (\gamma(t) - \tfrac{1}{2}\log z^2)\cdot z^2$$

that appeared in Eq. (151) into Eq. (145′) and paying attention to the sign of the constant of integration, we get the following expressions for

the function $\gamma(t)$:

$$\gamma_1 = \tfrac{3}{2} - \log\Big[\sqrt{\epsilon}\sinh 2\,\mathrm{artanh}\,\exp\big(\pm\sqrt{\epsilon}(t-t_0)\big)\Big],$$
$$\gamma_2 = \tfrac{3}{2} + \log\big[\pm t - t_0\big],$$
$$\gamma_3 = \tfrac{3}{2} - \log\Big[\sqrt{\epsilon}\cosh\log\tan\big(\sqrt{\epsilon}(\pm t - t_0)\big)\Big],$$

where $\epsilon > 0$ and $t_0 \in \mathbb{R}$. Now we return to formula (148) and assign the solution class

$$u = \tfrac{1}{2}z^2\log f(x)g(y) + (\gamma_i(t) - \tfrac{1}{2}\log z^2)\cdot z^2, \quad i = 1,2,3, \tag{152}$$

of the dispersionless Toda equation (144a) to each of the functions $\Phi(t,\,z)$.

The symmetry ϕ_4. Consider the constraint $\Phi_z = 0$; then, a solution to the equation $\Phi_{tt} = 1$ is a polynomial

$$\Phi(t) = \tfrac{1}{2}t^2 + C_1 t + C_2,$$

where C_1 and C_2 are constants of integration; the corresponding solutions of the dispersionless Toda equation (144a) are again defined in Eq. (148).

The symmetry $\varphi_2 \in \mathrm{Sym}\,\hat{\mathcal{E}}$. Now we construct the φ_2-invariant solutions of Eq. (144a). First, we consider the auxiliary ordinary differential equation in Eq. (151); then, we obtain the hyperbolic scal$^+$-Liouville equation (see Eq. (39) on page 30). Now, subsitute the expression $(\gamma(x,\,y) - \frac{1}{2}\log z^2)\cdot z^2$ for u in the equation $u_{xy} = \exp(-u_{zz})$ and make the transformation

$$\mathcal{X} = x\,\exp(3/2), \quad \mathcal{Y} = y\,\exp(3/2).$$

Hence we get the Liouville equation

$$\gamma_{\mathcal{X}\mathcal{Y}} = \exp(-2\gamma)$$

whose solutions ([69]) are easily obtained from formula (87) on page 49 by a trivial coordinates change; we have

$$\gamma(\mathcal{X},\mathcal{Y}) = -\tfrac{1}{2}\log\Big[-f'(\mathcal{X})g'(\mathcal{Y})\Big\{\mathbf{Q}\big([f(\mathcal{X})+g(\mathcal{Y})]^2\big)\Big\}^{-2}\Big],$$

where the mapping $\mathbf{Q}$ is one of sin, id, or sinh, whence we finally obtain

$$u(x,y,z) = \frac{z^2}{2}\log\frac{\Big\{\mathbf{Q}\big([f(\mathrm{e}^{3/2}x)+g(\mathrm{e}^{3/2}y)]^2\big)\Big\}^2}{-z^2 f'(\mathrm{e}^{3/2}x)g'(\mathrm{e}^{3/2}y)}. \tag{153}$$

This class of solutions of Eq. (144a) is in fact present in paper [2], and their physical interpretation is known (*ibid.*): these expressions provide the instanton solutions ([11]) of the antiselfdual vacuum Einstein equations (ASDVEE).

The symmetry $\varphi_3 \in \operatorname{Sym}\hat{\mathcal{E}}$. In order to obtain the φ_3-invariant solution of Eq. (144a), we solve the equations $u_z = 0$ and $u_{xy} = \exp(-u_{zz})$ together. The solution is

$$u(x, y, z) = xy + f(x) + g(y),$$

where f and g are arbitrary functions.

The symmetries φ_4 and φ_5 of Eq. (144a) depend neither on the unknow function u nor its derivatives. Therefore, if the equation $u_{xy} = \exp(-u_{zz})$ is overdetermined by the condition $\varphi_4 = 0$ or $\varphi_5 = 0$, then the problem of search for exact solutions to the dispersionless Toda equation is not simplified.

9.2. **The Noether symmetries and conservation laws.** In this subsection, we consider the problem of constructing the conservation laws for the dispersionless Toda equations (144) and (146). First, we discuss the general correlation between an Euler equation (15) and the fixed set of conservation laws which reflect the conservation properties of the energy–momentum tensor.

9.2.1. *On the Lagrangian formalism involving higher–order derivatives.* Suppose that $\mathcal{L} = \int L\, \mathrm{d}\boldsymbol{x}$ is a Lagrangian with the second order derivatives and $g_{\mu\nu}$ is the corresponding metric; then we can construct a conservative analog of the energy–momentum tensor $T^{\nu\mu}$ that satisfies the equation

$$\sum_{\mu} \bar{D}_\mu(T^{\nu\mu}) = 0, \qquad \bar{D}_\mu \equiv D_\mu\big|_{\mathcal{E}}. \tag{154}$$

The equations of motion $\mathcal{E}$ are

$$\sum_{\mu,\nu} D_\mu D_\nu \left(\frac{\partial L}{\partial u^i_{;\mu;\nu}}\right) = \sum_{\nu} D_\nu \left(\frac{\partial L}{\partial u^i_{;\nu}}\right) - \frac{\partial L}{\partial u^i} \tag{155}$$

and the tensor $T^{\nu\mu}$ coincides with its classical definition ([9]) if the higher–order term

$$\left\| \frac{\partial L}{\partial u^a_{;\mu;\nu}} \right\|$$

is trivial. Again, we use the notation $u^i_{;\mu} \equiv D_{x^\mu}(u^i)$ and $u^i_{;\mu;\nu} \equiv D_{x^\mu} \cdot D_{x^\nu}(u^i)$. We raise and lower the indexes $;\mu$, $;\nu$ by using the metric $g_{\mu\nu}$. By a straighforward substitution into Eq. (154) one checks that the tensor

$$T^{\nu\mu} = -g^{\nu\mu} L + \sum_i \frac{\partial L}{\partial u^i_{;\mu}}\, u^{i;\nu} + \sum_{i,\lambda} \Big[\frac{\partial L}{\partial u^i_{;\mu;\lambda}} \cdot D_\lambda\left(u^{i;\nu}\right) - D_\lambda\left(\frac{\partial L}{\partial u^i_{;\mu;\lambda}}\right) \cdot u^{i;\nu}\Big]$$

satisfies the required restrictions ([50]), and therefore condition (154) provides conservation laws for system (155). Still, we note an important feature of Eq. (146): as $r \to \infty$ and $\varepsilon \to +0$ in the Toda equations, the metric $g_{\mu\nu}$ becomes degenerate and the Lagrangian $\hat{\mathcal{L}}$ looses its

covariance. Indeed, it is impossible to find a nondegenerate metric $\hat{g}_{\mu\nu}$ such that

$$Ł \sim \tfrac{1}{8} u_{;\mu}{}^{;z} u^{;\mu}{}_{;z} + \exp(-u_{;z}^{;z}).$$

For the same reason we cannot define the energy–momentum tensor $T^{\nu\mu}$ for the equation $\mathcal{E}_{\text{heav}}$ as the variation of the Lagrangian $\hat{\mathcal{L}}_{\text{heav}}$ with respect to the metric $\hat{g}_{\nu\mu}$ since the nondegenerate metric does not exist.

9.2.2. *Constructing the conservation laws.* From the preceding section it is clear that the search of conservation laws for Eq. (146) is less trivial than finding the symmetry Lie fields by their generating functions and requires additional reasonings. In the sequel, we find the symmetries $\operatorname{sym} \mathcal{E}_{\text{heav}}$ of the dispersionless Toda equation, select its Noether symmetries $\operatorname{sym} \hat{\mathcal{L}}_{\text{heav}}$ in a fixed coordinate system, and reconstruct the conservation laws by using the homotopy method, which is described in [10, 94] and [45].

Lemma 56 ([50])**.** *The generating functions* (147a)*,* (147c)–(147e) *are the* Noether *symmetries of the equation* $\mathcal{E}_{\text{heav}}$*, while* (147b) *is not.*

Proposition 57 ([50])**.** *The conserved currents $\eta_i \in \bar{\Lambda}^2(\mathcal{E}_{\text{heav}})$, assigned to the Noether symmetries φ_i of Eq.* (146)*, are*

$$
\begin{aligned}
\eta_1 = \Big\{ &\frac{3}{8} f(x)\, u u_{xyzz} + \frac{1}{12} f(x)\, u_z u_{xyz} - \frac{1}{24} f(x)\, u_{xy} u_{zz} + \frac{1}{24} f(x)\, u_y u_{xzz} \\
&- \frac{1}{12} f'(x)\, u_y - \frac{1}{12}\, f(x) u_{xz} u_{yz} + \frac{z}{6} f'(x)\, u_{yz} + \frac{1}{8} f(x) u_x u_{yzz} \\
&- \frac{z^2}{8} f'(x)\, u_{yzz} - f(x) \int_0^1 t^2 u u_{zzz}^2 \exp(-t u_{zz})\, \mathrm{d}t \\
&\qquad + f(x) \int_0^1 t u u_{zzzz} \exp(-t u_{zz})\, \mathrm{d}t \Big\}\, \mathrm{d}y \wedge \mathrm{d}z \\
&+ \Big\{ -\frac{1}{8} f'(x)\, u u_{xzz} - \frac{1}{8} f(x)\, u u_{xxzz} + \frac{1}{4} f''(x)\, u + \frac{1}{12} f'(x)\, u_z u_{xz} \\
&+ \frac{1}{12} f(x)\, u_z u_{xxz} - \frac{z}{6} f''(z)\, u_z - \frac{1}{24} f'(x)\, u_x u_{zz} - \frac{1}{24} f(x)\, u_{xx} u_{zz} \\
&+ \frac{z^2}{24} f''(x)\, u_{zz} + \frac{1}{6} f(x)\, u_x u_{xzz} - \frac{1}{12} f'(x)\, u_x - \frac{1}{12} f(x)\, u_{xz}^2 \\
&\qquad + \frac{z}{6} f'(x)\, u_{xz} - \frac{z^2}{8} f'(x)\, u_{xzz} \Big\}\, \mathrm{d}z \wedge \mathrm{d}x \\
&+ \Big\{ -\frac{1}{6} f(x)\, u_{xy} u_{xz} + \frac{1}{3} f(x)\, u_x u_{xyz} - \frac{1}{12} f'(x)\, u_x u_{yz} - \frac{1}{4} f'(x)\, u u_{xyz} \\
&- \frac{z}{6} f''(x)\, u_y - \frac{z^2}{4} f'(x)\, u_{xyz} + \frac{1}{12} f(x)\, u_y u_{xxz} - \frac{1}{4} f(x)\, u u_{xxyz} \\
&+ \frac{1}{12} f(x)\, u_z u_{xxy} - \frac{1}{12} f(x)\, u_{xx} u_{yz} + \frac{z}{6} f'(x)\, u_{xy} + \frac{1}{12} f'(x)\, u_y u_{xz} \\
&+ \frac{1}{12} f'(x)\, u_z u_{xy} + \frac{z^2}{12} f''(x)\, u_{yz} + f(x) \cdot \int_0^1 \big[-t u u_{xzzz} - t u_{xz} u_{zz} + t u_x u_{zzz} \\
&+ t u_z u_{xzz} + t^2 u u_{xzz} u_{zzz} - t^2 u_x u_{zz} u_{zzz} \big] \exp(-t u_{zz})\, \mathrm{d}t + f'(x) \cdot \int_0^1 \big[-u_z \\
&\qquad + z u_{zz} - \frac{z^2}{2} u_{zzz} - t u u_{zzz} + \frac{z^2}{2} t u_{zz} u_{zzz} \big] \exp(-t u_{zz})\, \mathrm{d}t \Big\}\, \mathrm{d}x \wedge \mathrm{d}y,
\end{aligned}
$$

$$
\begin{aligned}
\eta_3 = &\left\{ \frac{5}{24} u_z u_{yzz} - \frac{1}{8} u_{yz} u_{zz} + \frac{1}{24} u_y u_{zzz} - \frac{1}{8} u u_{yzzz} \right\} \mathrm{d}y \wedge \mathrm{d}z \\
&+ \left\{ \frac{5}{24} u_z u_{xzz} - \frac{1}{8} u_{xz} u_{zz} + \frac{1}{24} u_x u_{zzz} - \frac{1}{8} u u_{xzzz} \right\} \mathrm{d}z \wedge \mathrm{d}x \\
&+ \Big\{ \frac{1}{4} u u_{xyzz} + \frac{1}{3} u_z u_{xyz} - \frac{1}{12} u_{xy} u_{zz} - \frac{1}{6} u_{xz} u_{yz} + \frac{1}{12} u_x u_{yzz} + \frac{1}{12} u_y u_{xzz} \\
&\qquad + u_{zz} \exp(-u_{zz}) + \exp(-u_{zz}) + u_z u_{zzz} \exp(-u_{zz}) \Big\}\, \mathrm{d}x \wedge \mathrm{d}y,
\end{aligned}
$$

$$
\begin{aligned}
\eta_4 = {} & \left\{ -\frac{1}{6}q(x)\,u_{yz} + \frac{z}{4}q(x)\,u_{yzz} \right\} \mathrm{d}y \wedge \mathrm{d}z \\
& + \left\{ \frac{1}{6}q'(x)\,u_z - \frac{z}{12}q'(x)\,u_{zz} - \frac{1}{6}q(x)\,u_{xz} + \frac{z}{4}q(x)\,u_{xzz} \right\} \mathrm{d}z \wedge \mathrm{d}x \\
& + \left\{ \frac{z}{2}q(x)\,u_{xyz} - \frac{1}{6}q(x)\,u_{xy} - \frac{z}{6}q'(x)\,u_{yz} + \frac{1}{6}q'(x)\,u_y \right. \\
& \qquad \left. - zq(x)\,u_{zzz}\exp(-u_{zz}) + q(x)\,\exp(-u_{zz}) \right\} \mathrm{d}x \wedge \mathrm{d}y,
\end{aligned}
$$

$$
\begin{aligned}
\eta_5 = {} & \frac{r(x)}{4}\,u_{yzz}\,\mathrm{d}y \wedge \mathrm{d}z + \left\{ -\frac{r'(x)}{12}\,u_{zz} + \frac{r(x)}{4}\,u_{xzz} \right\} \mathrm{d}z \wedge \mathrm{d}x \\
& + \left\{ -\frac{r'(x)}{6}\,u_{yz} + \frac{r(x)}{2}\,u_{xyz} + r(x)\,u_{zzz}\exp(-u_{zz}) \right\} \mathrm{d}x \wedge \mathrm{d}y.
\end{aligned}
$$

Any divergence $\bar{\mathrm{d}}_h(\eta_i)$ *equals* 0 *on the equation* $\mathcal{E}_{\mathrm{heav}}$; *calculating the divergences does not require precise evaluation of the integrals in the homotopy parameter* t *in rational functions, because differentiating with respect to* x *or* z *under the integral sign is allowed.*

Proving Lemma 56 and Proposition 57 is based on Lemma 48 and involves the general scheme of reconstruction of conservation laws by their generating sections.

Method of reconstruction of conserved currents. Now we describe a method ([94]) for constructing the differential $(n-1)$-forms that are exact on the equation $\mathcal{E} = \{F = 0\}$; we follow the paper [45].

Consider the function $f = u/\tau$ and the evolutionary vector field ∂_f whose flow is

$$A_\tau \colon (x^i, u^j_\sigma) \mapsto (x^i, \tau u^j_\sigma),$$

then we have

$$\frac{dA^*_\tau(\omega)}{d\tau} = A^*_\tau\left(\partial_f(\omega)\right) = A^*_\tau\left(\ell_\omega(f)\right) \tag{156}$$

for any differential form ω. Let this form ω be

$$\omega = \langle F, \psi \rangle = F\psi\, dx^1 \wedge \ldots \wedge dx^n$$

and suppose that η is the desired current which is assigned to the generating function ψ on the equation $\{F = 0\}$: $\mathrm{d}_h(\eta) = \nabla(F)$ and $\psi = \nabla^*(1)$. We note that the right-hand side in Eq. (156) contains the term

$$\ell_\omega(f) = \langle \ell^*_\omega(1), f \rangle + \mathrm{d}_h G(\ell_\omega \circ f),$$

where the first summand is cohomological to 0 since the image of the Euler operator is trivial if the Lagrangian density is a total divergence,

$$\ell^*_\omega(1) = \ell^*_{\langle F,\psi\rangle}(1) = \mathbf{E}(\langle F, \psi \rangle) = \mathbf{E}(\mathrm{d}_h\eta) = 0.$$

Next, the mapping $G\colon \mathcal{C}\mathrm{Diff}(\mathcal{F},\bar{\Lambda}^n) \to \bar{\Lambda}^{n-1}$, which is restricted onto the equation $\mathcal{E}$, is defined by the rule

$$G\Bigl(\sum_\sigma a_\sigma D_\sigma\Bigr) = \sum_{|\sigma|>0}\sum_{j\in\sigma} \frac{(-1)^{|\sigma|-1}}{|\sigma|} D_{\sigma-1_j}(a_\sigma)\omega_{(-j)},$$

where

$$\omega_{(-j)} = (-1)^{j+1}\,\mathrm{d}x^1\wedge\ldots\widehat{\mathrm{d}x^j}\ldots\wedge\mathrm{d}x^n$$

and $\sigma - 1_j$ is the result of excluding an index j from a multiindex $\sigma = (\sigma_1, \ldots, \sigma_n)$.

Now we integrate Eq. (156) in τ from 0 to 1 and get

$$\int_0^1 \frac{d}{d\tau}A_\tau^*(\omega)\,d\tau = A_1^*(\omega) - A_0^*(\omega) = \langle F,\psi\rangle - A_0^*\left(\langle F,\psi\rangle\right) =$$
$$= \int_0^1 A_\tau^*\left(\ell_\omega(f)\right)d\tau = \mathrm{d}_h\int_0^1 A_\tau^*\left(G(\ell_\omega\circ f)\right)d\tau = \mathrm{d}_h\eta,$$

whence we obtain the required formula for the current η.

Proof of Lemma 56 *and Proposition* 57. First we note that symmetries (147) of Eq. (144a) are also symmetries of the Euler equation (146): $\mathrm{Sym}\,\hat{\mathcal{E}} \subset \mathrm{sym}\,\mathcal{E}_{\mathrm{heav}}$, because the relation $\ell_{F_{\mathrm{heav}}} = D_z^2\circ\ell_{\hat{F}}$ holds due to Lemma 7 on page 18. By using Lemma 48, we check that symmetries (147a) and (147c)–(147e) of Eq. (146) are Noether, while for symmetry (147b) we have

$$\ell_{F_{\mathrm{heav}}}(\varphi_2) + \ell^*_{\varphi_2}(F_{\mathrm{heav}}) = 2u_{xyzz} - \tfrac{3}{2}D_z^2(\exp(-u_{zz})) \neq 0.$$

From Theorem 6 it follows that the Noether symmetries of Eq. (146) are in one-to-one correspondence with the generating sections of its conservation laws. By Corollary 10 on page 36, this correspondence for the equation $\mathcal{E}_{\mathrm{heav}}$ is provided by the identity mapping. Following the reasonings scheme above, to each of four Noether symmetries φ_i we assign the conserved current η_i; the arising expressions are rather large and therefore omitted. Nevertheless, the result is correct, for the conditions $\bar{\mathrm{d}}_h(\eta_i) = 0$ may be checked straightforwardly. $\square$

Remark 24. Suppose that the arbitrary function f in the conserved current η_1 is $f\equiv 1$. Then η_1 is a countinuous in z analog of component (58) of the energy–momentum tensor $\Theta = T\,\mathrm{d}x + \bar{T}\,\mathrm{d}y$ for the Toda equations (55). We recall that the conserved current $T\,\mathrm{d}x$ for Eq. (55) corresponds to the Noether symmetry $\varphi_0^1 = \square(\mathbf{E}_T(T\,\mathrm{d}x)) \in \mathrm{sym}\,\mathcal{L}_{\mathrm{Toda}}$. Another (nonlocal) conserved current

$$\eta = u_{xz}\exp(-u_{zz})\,\mathrm{d}x\wedge\mathrm{d}y + \left(\tfrac{1}{2}u_{xz}^2 - u_{xx}\right)\mathrm{d}x\wedge\mathrm{d}z,$$

for Eq. (146) (see Eq. (11) on page 12) could be also treated as an analog of the integral (58) for Eq. (55) due to the following reason.

Consider the Hamiltonian representation

$$u_y = D_x^{-1} \circ D_z^{-2} \circ \mathbf{E}_u \bigl(H_{\text{heav}} \, \mathrm{d}x \wedge \mathrm{d}z \bigr), \tag{157}$$

of Eq. (146), here

$$H_{\text{heav}} = -\exp(-u_{zz})$$

is the continuous limit of the Hamiltonian, see Eq. (117), for the Toda equations. By Lemma 44, the density H_{heav} is conserved on the related Hamiltonian equation (157), therefore the current η is also conserved.

Chapter 4. Bäcklund transformations and zero–curvature representations

In this chapter, we investigate the relationship between Bäcklund transformations and zero–curvature representations for the hyperbolic Liouville equation, the wave equation, and the Liouville scal$^+$-equation. They are

$$u_{xy} = \exp(2u), \qquad v_{xy} = 0, \qquad \Upsilon_{xy} = \exp(-2\Upsilon), \tag{158}$$

respectively. In Sec. 1 on page 29 we we discussed a natural geometric scheme that provides Eqs. (158) and gives their interpretation. In what follows, we construct one–parametric families of Bäcklund transformations and zero–curvature representations and consider exmaples of integrating Bäcklund transformations and zero–curvature in nonlocal variables.

10. Bäcklund transformations and their deformations

In this section, we study the properties of one–parametric deformations of Bäcklund transformations for Eq. (158). Consider the covering $\tau_t\colon \tilde{\mathcal{E}}_t \to \mathcal{E}_{\text{Liou}}$ defined by the extended total derivatives

$$\tilde{D}_x = \bar{D}_x + \tilde{u}_x \frac{\partial}{\partial \tilde{u}}, \quad \tilde{D}_y = \bar{D}_y + \tilde{u}_y \frac{\partial}{\partial \tilde{u}}, \qquad [\tilde{D}_x, \tilde{D}_y] = 0. \tag{159}$$

We assume that the partial derivatives of the nonlocal variable $\tilde{u}$ with respect to x and y are

$$\tilde{u}_x = u_x + \exp(-t) \cdot \exp(\tilde{u} + u), \tag{160a}$$

$$\tilde{u}_y = -u_y + 2\exp(t) \cdot \sinh(\tilde{u} - u), \tag{160b}$$

respectively. Recalling Remark 3 on page 19, let the diffeomorphism μ be the swapping $u \leftrightarrow \tilde{u}$ of the fiber variable u and the nonlocal variable $\tilde{u}$ combined with the mapping $x \mapsto -x$ and $y \mapsto -y$. Then diagram (25) supplies Bäcklund autotransformation $\mathcal{B}(\tilde{\mathcal{E}}_t,\ \tau_t,\ \tau_t \circ \mu,\ \mathcal{E}_{\text{Liou}})$ for Eq. (187), and for the Liouville equation. The equations $\tilde{\mathcal{E}}_t$ of this Bäcklund autotransformation ([14]) are, obviously,

$$(\tilde{u} - u)_x = \exp(-t) \cdot \exp(\tilde{u} + u), \tag{161a}$$

$$(\tilde{u} + u)_y = 2\exp(t) \cdot \sinh(\tilde{u} - u). \tag{161b}$$

Put

$$u_k \equiv \frac{\partial^k u}{\partial x^k} \text{ and } u_{\overline{k}} \equiv \frac{\partial^k u}{\partial y^k}$$

for any $k \in \mathbb{N}$. Consider the scaling symmetry

$$X^0 = -x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$$

of the Liouville equation and extend X^0 onto entire $\mathcal{E}^\infty_{\text{Liou}}$:

$$\hat{X} = -x\,\frac{\partial}{\partial x} + y\,\frac{\partial}{\partial y} + \sum_{k\geq 1} ku_k\,\frac{\partial}{\partial u_k} - \sum_{k\geq 1} ku_{\overline{k}}\,\frac{\partial}{\partial u_{\overline{k}}}. \tag{162}$$

Proposition 58. *The symmetry $\hat{X}$ cannot be extended to a symmetry of the covering equation $\tilde{\mathcal{E}}_t$.*

Proof. Assume the converse. By $\tilde{\partial}_\varphi$ denote the evolutionary vector field

$$\sum_\sigma \tilde{D}_\sigma(\varphi)\cdot\frac{\partial}{\partial u_\sigma}$$

on $\tilde{\mathcal{E}}_t$, where $\varphi\in C^\infty(\tilde{\mathcal{E}}_t)$. By definition, set

$$\tilde{\ell}_F(\varphi) = \tilde{\partial}_\varphi(F).$$

Suppose there is a smooth function $a\in C^\infty(\tilde{\mathcal{E}}_t)$ such that the linearized system

$$\tilde{\ell}_F(\varphi)=0,\qquad \tilde{D}_{x^i}(a) = \tilde{\partial}_{\varphi,a}(\tilde{u}_{x^i}) \equiv \left(\tilde{\partial}_\varphi + a\,\frac{\partial}{\partial\tilde{u}}\right)(\tilde{u}_{x^i}) \tag{163}$$

holds. This means that the field $\tilde{\partial}_{\varphi,a}$ is a local symmetry of the covering equation $\tilde{\mathcal{E}}_t$ and $\hat{X}$ is extended onto $\tilde{\mathcal{E}}_t$ constructively. Nevertheless, system (163) is not compatible since

$$\tilde{D}_x\circ\tilde{D}_y(a)\neq\tilde{D}_y\circ\tilde{D}_x(a).$$

Indeed, $\tilde{D}_x\circ\tilde{D}_y(a)-\tilde{D}_y\circ\tilde{D}_x(a)$ does not depend on a at all and equals

$$\begin{gathered} xu_x^2\mathrm{e}^{t+u-\tilde{u}} + u_xyu_y\mathrm{e}^{t+\tilde{u}-u} - xu_x\mathrm{e}^{2\tilde{u}} - u_xyu_y\mathrm{e}^{t+u-\tilde{u}} \\ -2yu_y\mathrm{e}^{2t+\tilde{u}+u} + 2xu_x\mathrm{e}^{2t+\tilde{u}+u} - xu_x^2\mathrm{e}^{t+\tilde{u}-u} + x\mathrm{e}^{2u}u_x \\ -y\mathrm{e}^{2u}u_y + 2\mathrm{e}^t xu_x^2 + yu_y\mathrm{e}^{2\tilde{u}} - 2\mathrm{e}^t yu_yu_x \neq 0. \end{gathered}$$

This contradiction concludes the proof. □

Therefore, the scaling symmetry $\hat{X}$ is a τ_t-*shadow* only (*i.e.*, a solution to the equation $\tilde{\ell}_F(\varphi)=0$) and generates the family of the covering equations $\tilde{\mathcal{E}}_t$ over $\mathcal{E}_{\text{Liou}}$; these $\tilde{\mathcal{E}}_t$ are parametrized by $t\in\mathbb{R}$.

Consider the Cartan distribution $\tilde{\mathcal{C}}_t$ on the equation $\tilde{\mathcal{E}}_t$ and by U_t denote the Cartan's connection form on $\tilde{\mathcal{E}}_t$. In local coordinates, the form U_t is

$$\begin{aligned} U_t = \sum_\sigma \mathrm{d}_{\mathcal{C}}(u_\sigma)\otimes\frac{\partial}{\partial u_\sigma} + \Big(\mathrm{d}\tilde{u} - \big(u_x + \exp(\tilde{u}+u-t)\big)\,\mathrm{d}x + {}\\ + \big(u_y - 2\exp(t)\sinh(\tilde{u}-u)\big)\,\mathrm{d}y\Big)\otimes\frac{\partial}{\partial\tilde{u}}, \end{aligned} \tag{164}$$

where $\mathrm{d}_{\mathcal{C}}$ is the Cartan differential. In coordinates, we have
$$\mathrm{d}_{\mathcal{C}}(u_\sigma) = \mathrm{d}u_\sigma - \sum_i \bar{D}_i(u_\sigma)\,\mathrm{d}x^i.$$

Let $\Omega \in \mathrm{D}(\Lambda^\mu(\tilde{\mathcal{E}}))$, then μ is the *degree* of the derivation Ω. By $[\![\cdot,\cdot]\!]^{\mathrm{FN}}$ we denote the *Frölicher–Nijenhuis bracket* ([37, 62]):
$$[\![\Omega,\Theta]\!]^{\mathrm{FN}}(f) = \mathrm{L}_\Omega(\Theta(f)) - (-1)^{\mu\nu}\cdot \mathrm{L}_\Theta(\Omega(f)), \tag{165}$$
where $\Omega,\Theta \in \mathrm{D}(\Lambda^*(\mathcal{E}))$, $f \in C^\infty(\mathcal{E})$, and the degrees μ, ν are $\mu = \deg\Omega$, $\nu = \deg\Theta$, respectively. We also assume that
$$\mathrm{L}_\Omega = [i_\Omega, \mathrm{d}]\colon \Lambda^k(\mathcal{E}) \to \Lambda^{k+\deg\Omega}(\mathcal{E})$$
is the Lie derivative and
$$i_\Omega\colon \Lambda^k(\mathcal{E}) \to \Lambda^{k+\deg\Omega-1}(\mathcal{E})$$
is the inner product. The Frölicher–Nijenhuis bracket is a natural geometric structure in differential calculus. The de Rham differential d and the Richardson–Nijenhuis bracket $[\![\cdot,\cdot]\!]^{\mathrm{RN}}$ (see [52] for its applications) are other examples of natural structures.

Theorem 59 ([37])**.** *Let $\tau\colon \tilde{\mathcal{E}} \to \mathcal{E}$ be a covering and $A_t\colon \tilde{\mathcal{E}} \to \tilde{\mathcal{E}}$ be a smooth family of diffeomorphisms such that $A_0 = \mathrm{id}$ and $\tau_t = \tau\circ A_t\colon \tilde{\mathcal{E}} \to \mathcal{E}$ is the covering for any $t \in \mathbb{R}$. Then the evolution of the Cartan connection form U_{τ_t} is*
$$\frac{dU_{\tau_t}}{dt} = [\![\hat{X}_t, U_{\tau_t}]\!]^{\mathrm{FN}}, \tag{166}$$
where $\hat{X}_t$ is a τ_t-shadow for any $t \in \mathbb{R}$.

Suppose $\tilde{\mathcal{E}}$ is a finite-dimensional manifold, then there is the isomorphism
$$\mathrm{D}(\Lambda^*(\tilde{\mathcal{E}})) \simeq \Lambda^*(\tilde{\mathcal{E}}) \otimes \mathrm{D}(\tilde{\mathcal{E}}).$$
Thence, any derivation $\Omega \in \mathrm{D}(\Lambda^*(\tilde{\mathcal{E}}))$ is decomposable to the finite sum such that the summands are $\Omega = \omega\otimes X$, where $\omega \in \Lambda^*(\tilde{\mathcal{E}})$ and $X \in \mathrm{D}(\tilde{\mathcal{E}})$. The Frölicher–Nijenhuis bracket of such elements is
$$\begin{aligned}[\![\omega\otimes X, \theta\otimes Y]\!]^{\mathrm{FN}} &= \omega\wedge\theta\otimes[X,Y] + \omega\wedge \mathrm{L}_X(\theta)\otimes(Y) + \\ &+ (-1)^i\,\mathrm{d}\omega\wedge(X \lrcorner\, \theta)\otimes Y - (-1)^{ij}\,\theta\wedge\mathrm{L}_Y(\omega)\otimes X - \\ &- (-1)^{(i+1)j}\,\mathrm{d}\theta\wedge(Y \lrcorner\, \omega)\otimes X,\end{aligned} \tag{167}$$
where X, $Y \in \mathrm{D}(\tilde{\mathcal{E}})$, $\omega \in \Lambda^i(\tilde{\mathcal{E}})$ and $\theta \in \Lambda^j(\tilde{\mathcal{E}})$. If the dimension of $\tilde{\mathcal{E}}$ is not necessarily finite, then there is the embedding
$$\Lambda^*(\tilde{\mathcal{E}})\otimes\mathrm{D}(\tilde{\mathcal{E}}) \subset \mathrm{D}(\Lambda^*(\tilde{\mathcal{E}}))$$
defined by the rule
$$(\omega\otimes X)(f) = X(f)\omega$$
for any function $f \in C^\infty(\tilde{\mathcal{E}})$.

Consider the coverings $\tau_t\colon \tilde{\mathcal{E}}_t \to \mathcal{E}$ defined in Eq. (159). Then we have

$$\frac{dU_t}{dt} = (\exp(\tilde{u} + u - t)\,\mathrm{d}x - 2\exp(t)\sinh(\tilde{u} - u)\,\mathrm{d}y) \otimes \frac{\partial}{\partial \tilde{u}}. \tag{168}$$

We claim that the scaling symmetry $\hat{X}$ is the τ_t-shadow such that the evolution of the connection form U_t (164) in the covering τ_t, see Eq. (159), is given by (168) in virtue of equation (166). We need Lemmas 60-65 to prove this.

Lemma 60. $[\![\hat{X}, U_t]\!]^{\mathrm{FN}} \lrcorner\, \mathrm{d}\tilde{u} = \big(dU_t/dt\big) \lrcorner\, \mathrm{d}\tilde{u}$.

Lemma 61. $[\![\hat{X}, U_t]\!]^{\mathrm{FN}} \lrcorner\, \mathrm{d}x = [\![\hat{X}, U_t]\!]^{\mathrm{FN}} \lrcorner\, \mathrm{d}y = [\![\hat{X}, U_t]\!]^{\mathrm{FN}} \lrcorner\, \mathrm{d}u = 0.$

Proof. The proof of Lemmas 60 and 61 is based on successive application of formula (167). The coefficient of $\partial/\partial x$ is

$$-\sum_\sigma \mathrm{d}_{\mathcal{C}} u_\sigma \wedge \mathrm{L}_{\partial/\partial u_\sigma}(-x) - \mathrm{d}_{\mathcal{C}}\tilde{u} \wedge \mathrm{L}_{\partial/\partial\tilde{u}}(-x) = 0;$$

the coefficient of $\partial/\partial y$ is calculated analogously. Now consider Eq. (162) and decompose $\hat{X}$ in the coordinates $\langle x,\ y,\ u_k,\ u_{\overline{k}}\rangle$. We get $\hat{X} = \sum_\alpha \omega_\alpha \otimes X_\alpha$, where ω_α is a 0-form and X_α is a derivation for any α. We have $i = 0$ and $j = 1$ in Eq. (167). Thence we get

$$\sum_\alpha \big(\omega_\alpha \wedge \mathrm{L}_{X_\alpha}(\mathrm{d}_{\mathcal{C}} u) + \mathrm{d}\omega_\alpha \wedge (X_\alpha \lrcorner\, \mathrm{d}_{\mathcal{C}} u)\big) \otimes \frac{\partial}{\partial u},$$

where the first summand is

$$\sum_\alpha \omega_\alpha \wedge \mathrm{d}(X_\alpha \lrcorner\, \mathrm{d}_{\mathcal{C}} u) + \sum_\alpha \omega_\alpha \wedge \big(X_\alpha \lrcorner\, \mathrm{d}(\mathrm{d}_{\mathcal{C}} u)\big) = -u_x\,\mathrm{d}x + u_y\,\mathrm{d}y,$$

and the second summand equals $u_x\,\mathrm{d}x - u_y\,\mathrm{d}y$, whence their sum is also trivial. Finally, we calculate the coefficient

$$\sum_\alpha \big(\omega_\alpha \wedge \mathrm{L}_{X_\alpha}(\mathrm{d}_{\mathcal{C}}\tilde{u}) + \mathrm{d}\omega_\alpha \wedge (X_\alpha \lrcorner\, \mathrm{d}_{\mathcal{C}}\tilde{u})\big)$$

of $\partial/\partial\tilde{u}$ by using the explicit formula for $\mathrm{d}_{\mathcal{C}}\tilde{u}$. The first summand is equal to $-u_x\,\mathrm{d}x - u_y\,\mathrm{d}y$, and the second summand is

$$\big(u_x + \mathrm{e}^{-t}\exp(\tilde{u} + u)\big)\,\mathrm{d}x + \big(u_y - 2\mathrm{e}^{t}\sinh(\tilde{u} - u)\big)\,\mathrm{d}y.$$

Consequently, we obtain the required expression

$$\big(\mathrm{e}^{-t}\exp(\tilde{u} + u)\,\mathrm{d}x - 2\mathrm{e}^{t}\sinh(\tilde{u} - u)\,\mathrm{d}y\big) \otimes \frac{\partial}{\partial\tilde{u}}.$$

This concludes the proof. □

However, the calculation of coefficients of $\partial/\partial u_k$ or $\partial/\partial u_{\overline{k}}$ in $[\![\hat{X}, U_t]\!]^{\mathrm{FN}}$ is untrivial for $k \geq 1$. First, we have

Lemma 62 ([47]). *Let $u(x)$ and $f(u)$ be smooth functions and D_x be the total derivative with respect to x. Denote $u_k \equiv D_x^k(u(x))$, $k \geq 0$, $u_0 \equiv u$. Then the relation*

$$n \cdot D_x^n(f(u)) = \sum_{m=1}^{n} m\, u_m \frac{\partial}{\partial u_m} D_x^n(f(u)) \tag{169}$$

holds for any integer $n \geq 1$.

The proof of Lemma 62 is based on Lemma 63 and Corollary 64 below.

Lemma 63. *Let the assumptions of Lemma 62 hold. Take an integer $n > 0$ and a positive integer $l \leq n-1$. Then the relation*

$$D_x\left(\frac{\partial}{\partial u_l} D_x^{n-1}(f(u))\right) = \frac{\partial}{\partial u_l} D_x^n(f(u)) - \frac{\partial}{\partial u_{l-1}} D_x^{n-1}(f(u)) \tag{170}$$

is valid.

Corollary 64. Let the assumptions of Lemma 63 hold. Then we also have

$$(n+1)u_{n+1} \frac{\partial}{\partial u_{n+1}} D_x^{n+1}(f(u)) = (n+1)u_{n+1} \frac{\partial}{\partial u_n} D_x^n(f(u)) = \\ = (n+1)u_{n+1} \cdot f'(u). \tag{171}$$

Proof of Lemma 62 ([47]). We prove (169) by induction on n. For $n=1$, relation (169) holds. For $n \geq 1$, we have

$$(n+1)\, D_x^{n+1}(f(u)) = D_x(n\, D_x^n(f(u)) + D_x^n(f(u))) =$$

by the inductive assumption,

$$= D_x\left(\sum_{m=1}^{n} m u_m \frac{\partial}{\partial u_m} D_x^n(f(u)) + D_x^n(f(u))\right) =$$

by the Leibniz rule,

$$= \sum_{m=1}^{n} m u_{m+1} \frac{\partial}{\partial u_m} D_x^n(f(u)) + \\ + \sum_{m=1}^{n} m u_m\, D_x \frac{\partial}{\partial u_m} D_x^n(f(u)) + D_x\, D_x^n(f(u)) =$$

by Eq. (170) applied to the second sum,

$$= \sum_{m=1}^{n} m u_m \frac{\partial}{\partial u_m} D_x^{n+1}(f(u)) + \sum_{m=1}^{n} m u_{m+1} \frac{\partial}{\partial u_m} D_x^n(f(u)) - \\ - \sum_{m=1}^{n} m u_m \frac{\partial}{\partial u_{m-1}} D_x^n(f(u)) + D_x D_x^n(f(u)) =$$

by the definition of D_x and the subscript shift in the latter sum,

$$= \sum_{m=1}^{n} mu_m \frac{\partial}{\partial u_m} D_x^{n+1}(f(u)) + \sum_{m=0}^{n}(m+1)u_{m+1} \frac{\partial}{\partial u_m} D_x^{n}(f(u)) - \\ - \sum_{m=0}^{n-1}(m+1)u_{m+1} \frac{\partial}{\partial u_m} D_x^{n}(f(u)) =$$

since almost all summands in the latter two sums coincide,

$$= \sum_{m=1}^{n} mu_m \frac{\partial}{\partial u_m} D_x^{n+1}(f(u)) + (n+1)u_{n+1} \frac{\partial}{\partial u_n} D_x^{n}(f(u)) =$$

by (171),

$$= \sum_{m=1}^{n} mu_m \frac{\partial}{\partial u_m} D_x^{n+1}(f(u)) + (n+1)u_{n+1} \frac{\partial}{\partial u_{n+1}} D_x^{n+1}(f(u)) = \\ = \sum_{m=1}^{n+1} mu_m \frac{\partial}{\partial u_m} D_x^{n+1}(f(u)).$$

The proof is complete. □

Here we offer another proof of Lemma 62. This proof is based on the weights technique. The idea of this proof was communicated by V. V. Trushkov.

Proof of Lemma 62 ([58]). By definition, put the *weight* $\mathrm{wt}(u_k) = k$, $\mathrm{wt}(u_k \cdot u_l) = k + l$, and $\mathrm{wt}(u_{k_1} + u_{k_2}) = k_1$ if $k_1 = k_2$. We have

$$D_x^n(f(u)) = \sum_{m=1}^{n} P_{n,m} \cdot f^{(m)}(u), \tag{172}$$

where $P_{n,m} = \sum_{\vec{\jmath}} \mathrm{const}(n,m) \cdot u_{j_1} \cdot \ldots \cdot u_{j_{l(n,m)}}$. We claim that $P_{n,m}$ is a differential polynomial such that

$$j_1 + \ldots + j_{l(n,m)} = n \qquad \forall \vec{\jmath}, \quad \forall\ n,\ m. \tag{173}$$

We prove this fact by induction on n. Indeed, if $\mathrm{wt}(P_{n,m}) = n$, then $\mathrm{wt}(D_x(P_{n,m}))) = n + 1$ due to the Leibnitz rule. Besides,

$$D_x^{n+1}(f(u)) = \sum_{m=1}^{n} \left(D_x(P_{n,m}) \cdot f^{(m)}(u) + P_{n,m} \cdot u_1 \cdot f^{(m+1)}(u)\right)$$

and therefore the weight $\mathrm{wt}(D_x^{n+1}(f(u)))$ is well defined and equals $n + 1$.

Now, consider the *weight counting operator*

$$\mathcal{W} \equiv \sum_{m\geq 1} m \cdot u_m \frac{\partial}{\partial u_m} \tag{174}$$

that acts on the right-hand side in Eq. (172):

$$\sum_{m\geq 1} m\cdot u_m\,\frac{\partial}{\partial u_m}\circ\sum_{k=1}^{n}\sum_{\vec{\jmath}}\mathrm{const}(n,k)\cdot u_{j_1}\cdot\ldots\cdot u_{j_{l(n,k)}}\cdot f^{(k)}(u)=$$
$$=\sum_{k=1}^{n}\sum_{\vec{\jmath}}\mathrm{const}(n,k)\cdot n\cdot u_{j_1}\cdot\ldots\cdot u_{j_{l(n,k)}}\cdot f^{(k)}(u)=n\cdot D_x^n(f(u)),$$

since condition (173) holds for all multiindexes $\vec{\jmath}$. We finally conclude that the functions $D_x^n(f(u))$ are eigenfunctions for operator (174) and the integers $n\in\mathbb{N}$ are the eigenvalues. Therefore, relation (169) is a solution to the problem $\lambda\cdot\varphi=\mathcal{W}(\varphi)$. The proof is complete. □

Lemma 65. $[\![\hat{X},U_t]\!]^{\mathrm{FN}}\rfloor\,\mathrm{d}u_k=[\![\hat{X},U_t]\!]^{\mathrm{FN}}\rfloor\,\mathrm{d}u_{\bar{k}}=0$ *for any* $k\geq 1$.

Proof. Let $k\in\mathbb{N}$. Taking into account Eq. (167), consider the 1-form

$$[\![\hat{X},U_t]\!]^{\mathrm{FN}}\rfloor\,\mathrm{d}u_k=\Bigl((k-1)\bar{D}_y u_k-\sum_{l=1}^{k-1} l u_l\,\frac{\partial}{\partial u_l}\bar{D}_x^{k-1}(\exp(2u))\Bigr)\cdot\mathrm{d}y.$$

We see that all coefficients of $\mathrm{d}x$, $\mathrm{d}u$, $\mathrm{d}u_l$, $\mathrm{d}u_{\bar{l}}$ vanish for all $l\geq 1$. Finally, note that $\bar{D}_y u_k=\bar{D}_x^{k-1}(\exp(2u))$. By Lemma 62, the coefficient of $\mathrm{d}y$ is trivial. Arguing as above, we conclude that $[\![\hat{X},U_t]\!]^{\mathrm{FN}}\rfloor\,du_{\bar{k}}=0$. This completes the proof. □

Theorem 66. *The* τ_t*-shadow in Eq.* (162) *satisfies the relation*

$$[\![\hat{X},U_t]\!]^{\mathrm{FN}}=(\exp(\tilde{u}+u-t)\,dx-2\exp(t)\sinh(\tilde{u}-u)\,dy)\otimes\frac{\partial}{\partial\tilde{u}},$$

i.e., the abelian group of diffeomorfisms $A_t=\exp(t\hat{X})$ *provides smooth one-parametric family* (160) *of one-dimensional coverings over Liouville's equation* $\mathcal{E}_{\mathrm{Liou}}$*. These coverings correspond to Bäcklund autotransformations* (161) *for the Liouville equation, see diagram* (25)*. Evolution of the connection form is given by* (168)*.*

Proof. Decompose the bracket $[\![\hat{X},U_t]\!]^{\mathrm{FN}}$ with respect to the basis $\langle\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial u_k}$, $\frac{\partial}{\partial u_{\bar{k}}}$, $\frac{\partial}{\partial\tilde{u}}\rangle$. By Lemmas 61 and 65, all coefficients of the derivations $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial u_k}$, $\frac{\partial}{\partial u_{\bar{k}}}$ equal 0 for $k\geq 0$. Thence, Lemma 60 supplies the required expression, which is given in Eq. (168). □

Remark 25. Consider Bäcklund transformation ([14])

$$(v-u)_x=\mathrm{e}^{-t}\exp(u+v),\ (v+u)_y=-\mathrm{e}^{t}\exp(u-v),\qquad(175)$$

$t\in\mathbb{R}$, between the Liouville equation (76) and the wave equation $v_{xy}=0$, as well as Bäcklund transformation ([14])

$$(\Upsilon-u)_x=2\mathrm{e}^{-t}\cosh(\Upsilon+u),\ (\Upsilon+u)_y=-\mathrm{e}^{t}\exp(u-\Upsilon),\qquad(176)$$

$t \in \mathbb{R}$, between $\mathcal{E}_{\text{Liou}}$ and the Liouville scal$^+$-equation $\Upsilon_{xy} = \exp(-2\Upsilon)$. Then these transformations possess the property similar to one described in Theorem 66. Namely, scaling symmetry (162) of $\mathcal{E}_{\text{Liou}}$ is the required τ_t-shadow in both cases. The proof of these theorems is quite analogous to the proof of Theorem 66 above and appeals to useful identity (169) in total derivatives.

11. Integrating Bäcklund transformations in nonlocal variables

In this section, we study nonlocal aspects of integrating Bäcklund transformations between PDE. The aim of this section is to illustrate a natural scheme that provides nonlocal variables associated with a certain PDE and to obtain nonlocal structures for the hyperbolic Liouville equation. We reconstruct the coverings τ_j from Eq. (161) and (175)–(176) and demonstrate the nonlocal variables to be potentials for the fiber variables u, v, and Υ. In what follows, we use the notation $\mathcal{E}_u$ as a synonim of $\mathcal{E}_{\text{Liou}}$, meaning that the Liouville equation $u_{xy} = \exp(2u)$ is imposed on the fiber variable u.

Now we construct the one-dimensional non-abelian coverings such that we can integrate Bäcklund transformations (161), (175)–(176) in the corresponding nonlocal variables. We fix an arbitrary $t \in \mathbb{R}$ and consider the extended total derivatives

$$\begin{aligned}
&\tilde{D}_x^{\mathcal{E}_u} = \bar{D}_x^{\mathcal{E}_u} - \mathrm{e}^{2u}\frac{\partial}{\partial \Xi_t}, &&\tilde{D}_y^{\mathcal{E}_u} = \bar{D}_y^{\mathcal{E}_u} + (\Xi_t^2 + 2u_y\Xi_t - \mathrm{e}^{2t})\frac{\partial}{\partial \Xi_t},\\
&\tilde{D}_x^{\mathcal{E}_u} = \bar{D}_x^{\mathcal{E}_u} - \mathrm{e}^{2u}\frac{\partial}{\partial \Xi_\infty}, &&\tilde{D}_y^{\mathcal{E}_u} = \bar{D}_y^{\mathcal{E}_u} + (\Xi_\infty^2 + 2u_y\Xi_\infty)\frac{\partial}{\partial \Xi_\infty},\\
&\tilde{D}_y^{\mathcal{E}_u} = \bar{D}_y^{\mathcal{E}_u} - \mathrm{e}^{2u}\frac{\partial}{\partial \Xi'_t}, &&\tilde{D}_x^{\mathcal{E}_u} = \bar{D}_x^{\mathcal{E}_u} + (\Xi_t'^2 + 2u_x\Xi'_t + \mathrm{e}^{-2t})\frac{\partial}{\partial \Xi'_t},\\
&\tilde{D}_x^{\mathcal{E}_v} = \bar{D}_x^{\mathcal{E}_v} + \mathrm{e}^{2v}\frac{\partial}{\partial \Xi_t^v}, &&\tilde{D}_y^{\mathcal{E}_v} = \bar{D}_y^{\mathcal{E}_v} + (2v_y\Xi_t^v + \mathrm{e}^{2t})\frac{\partial}{\partial \Xi_t^v},\\
&\tilde{D}_y^{\mathcal{E}_\Upsilon} = \bar{D}_y^{\mathcal{E}_\Upsilon} + \mathrm{e}^{-2\Upsilon}\frac{\partial}{\partial \Xi_t^\Upsilon}, &&\tilde{D}_x^{\mathcal{E}_\Upsilon} = \bar{D}_x^{\mathcal{E}_\Upsilon} + ((\Xi_t^\Upsilon)^2 - 2\Upsilon_x\Xi_t^\Upsilon + \mathrm{e}^{-2t})\frac{\partial}{\partial \Xi_t^\Upsilon}.
\end{aligned} \tag{177}$$

We see that the extended derivatives commute in all cases:

$$[\tilde{D}_x, \tilde{D}_y] = 0.$$

Therefore the coverings

$$\tau_t\colon \tilde{\mathcal{E}}_t \to \mathcal{E}_u^\infty, \quad \tau_\infty\colon \tilde{\mathcal{E}}_\infty \to \mathcal{E}_u^\infty, \quad \tau'_t\colon \tilde{\mathcal{E}}'_t \to \mathcal{E}_u^\infty,$$
$$\tau_t^v\colon \tilde{\mathcal{E}}_t^v \to \mathcal{E}_v^\infty, \quad \tau_t^\Upsilon\colon \tilde{\mathcal{E}}_t^\Upsilon \to \mathcal{E}_\Upsilon^\infty \tag{178}$$

are well defined. The explicit form of the covering equations $\tilde{\mathcal{E}}_t$, $\tilde{\mathcal{E}}_\infty$, $\tilde{\mathcal{E}}'_t$, $\tilde{\mathcal{E}}_t^v$, and $\tilde{\mathcal{E}}_t^\Upsilon$ is discussed in Remark 27.

Remark 26. Coverings (178) with nonlocal variables (177) cannot be reduced to local conservation laws for the underlying equations $\mathcal{E}_u$, $\mathcal{E}_v$, and $\mathcal{E}_\Upsilon$; these coverings are called *non–abelian*. We also claim that the t-parameterized coverings (*e.g.* τ_t at the points t_1 and t_2) are equivalent ($\tau_{t_1} \simeq \tau_{t_2}$), *i.e.*, there is a functional dependence between the nonlocal variables (Ξ_{t_1} and Ξ_{t_2} in our case). We have

$$\Xi_{t_1} + y \cdot \exp(2t_1) = \Xi_{t_2} + y \cdot \exp(2t_2) = \Xi_{t=-\infty} \qquad \forall t_1, t_2 \in \mathbb{R};$$

similar relations hold for other coverings (178).

Remark 27. The covering equations can be obtained explicitly since each nonlocal variable in (177) is a potential for at least one of the dependent variables u, v, and Υ. For example,

$$u = \frac{1}{2}\log\Bigl(-\frac{\partial \Xi_\infty}{\partial x}\Bigr).$$

Now we derive the covering equations imposed on the fiber variable Ξ_t and its limit Ξ_∞ at the point $t = -\infty$:

$$\tilde{\mathcal{E}}_t = \left\{ \frac{\partial \Xi_t}{\partial y} = \Xi_t^2 + \frac{\Xi_t \cdot \frac{\partial^2 \Xi_t}{\partial x \partial y}}{\frac{\partial \Xi_t}{\partial x}} - \exp(2t) \right\}, \tag{179}$$

where $t \in \mathbb{R} \cup \{-\infty\}$. The equations $\tilde{\mathcal{E}}_t'$, $\tilde{\mathcal{E}}_t^v$, and $\tilde{\mathcal{E}}_t^\Upsilon$ are obtained analogously.

11.1. **Integrating in nonlocal variables.** We emphasize that transformations (161), (175), and (176) cannot be integrated in local variables. Therefore we consider one-dimensional non-abelian coverings (178) and extend the sets of (local) variables with new nonlocal ones, see (177), and then we integrate the transformations successfully. The results can be summarized as follows.

Theorem 67 ([58])**.** *Consider Bäcklund* (*auto*)*transformations* (161), (175)–(176) *for the equations* $\mathcal{E}_u$, $\mathcal{E}_v$, *and* $\mathcal{E}_\Upsilon$. *These transformations are integrated in nonlocal variables explicitly*:

(1) *Bäcklund autotransformation* (161) *for* $\mathcal{E}_u$:

$$\tilde{u} = u + t - \log \Xi_t \quad u = t + \tilde{u} - \log \Xi_t[\tilde{u}](-x, -y),$$

i.e., to inverse the transformation and obtain $u[\tilde{u}]$, *the inversion* $x \mapsto -x$ *and* $y \mapsto -y$ *is required.*

(2) *Bäcklund transformation* (175) *between* $\mathcal{E}_u$ *and the wave equation* $v_{xy} = 0$:

$$v = u + t - \log \Xi_\infty \text{ and, conversely, } u = v + t - \log \Xi_t^v.$$

(3) *Bäcklund transformation* (176) *between* $\mathcal{E}_u$ *and the* scal$^+$*–Liouville equation* $\Upsilon_{xy} = \exp(-2\Upsilon)$:

$$\Upsilon = -u + t + \log \Xi_t' \text{ and, conversely, } u = -\Upsilon - t - \log \Xi_t^\Upsilon.$$

Proof. We consider the case $\tilde{u}[u](x,y)$ within Bäcklund autotransformation (161). By definition, put $\mathcal{U} = \exp(\tilde{u})$ and $\mathcal{T} = \exp(-\tilde{u})$. From Eq. (161) we obtain the Bernoulli equation

$$\mathcal{U}_x = u_x \cdot \mathcal{U} + \exp(u-t)\,\mathcal{U}^2,$$

whence $\mathcal{U}^{-1} = \mathcal{T} = \exp(-u-t)\cdot\Xi$, where the nonlocal variable Ξ is such that $\tilde{D}_x(\Xi) = -\exp(2u)$, and also get the Riccati equation

$$\mathcal{T}_y = u_y \cdot \mathcal{T} + \exp(u+t)\,\mathcal{T}^2 - \exp(t-u). \tag{180}$$

Substituting $\exp(-u-t)\cdot\Xi$ for $\mathcal{T}$ in (180), we get $\tilde{D}_y(\Xi) = \Xi^2 + 2u_y\Xi - \exp(2t)$. Now we refer (177) and compare the result with the derivatives $\tilde{D}_x(\Xi_t)$ and $\tilde{D}_y(\Xi_t)$.

The proof of other five cases is quite analogous. Assume $f(x,y)\in\{u, \tilde{u}, v, \Upsilon\}$ is a known solution to the PDE $\mathcal{E}_f$. Then we obtain either two Bernoulli equations for Eq. (175) or one Bernoulli equation and one Riccati equation for Eq. (161) and Eq. (176) after a proper change of variables. Solving these ordinary differential equations for the solution $g(x,\,y)\in\{u,\,\tilde{u},\,v,\,\Upsilon\}$ to the PDE $\mathcal{E}_g$ that is related with $\mathcal{E}_f$ by one of Bäcklund (auto)transformations (175)–(176), we finally obtain the rules to differentiate the nonlocal variable, which appears in one of the coverings (178). $\square$

Consider diagram (24) that arises in the definition of a Bäcklund transformation and apply Eq. (25) to Theorem 67. Consider the coverings τ_1 and τ_2 in Eq. (177). We see that in all cases one of these two mappings is a first order differential operator that depends on the nonlocal variable only, while another projector is a zero order morphism. Thence we have

Theorem 68. *Consider equations* (179)*. Then the following relations hold:*

$$\begin{aligned}
u &= \tfrac{1}{2}\log\left(-\Xi_t\right)_x, &\qquad \tilde{u} &= t + \log\bigl[\Xi_t^{-1}\sqrt{(-\Xi_t)_x}\bigr],\\
u &= \tfrac{1}{2}\log\left(-\Xi_\infty\right)_x, &\qquad v &= t + \log\bigl[\Xi_\infty^{-1}\sqrt{(-\Xi_\infty)_x}\bigr],\\
u &= \tfrac{1}{2}\log\left(-\Xi'_t\right)_y, &\qquad \Upsilon &= t + \log\Bigl[\frac{\Xi'_t}{\sqrt{(-\Xi'_t)_y}}\Bigr],\\
v &= \tfrac{1}{2}\log\left(\Xi_t^v\right)_x, &\qquad u &= t + \log\bigl[(\Xi_t^v)^{-1}\sqrt{(\Xi_t^v)_x}\bigr],\\
\Upsilon &= -\tfrac{1}{2}\log\left(\Xi_t^\Upsilon\right)_y, &\qquad u &= -t + \log\bigl[(\Xi_t^\Upsilon)^{-1}\sqrt{(\Xi_t^\Upsilon)_y}\bigr].
\end{aligned}$$

In other words, the nonlocal variables that satisfy Eq. (179) are potentials for *both* solutions $f,\ g$ to the equations $\mathcal{E}_f,\ \mathcal{E}_g$ within the list $\mathcal{E}_u$, $\mathcal{E}_v$, and $\mathcal{E}_\Upsilon$. The property of all the coverings in (178) to be nonlinear differential operators of order not greater than 1 is a special feature of these equations.

11.2. **On nonlocal symmetries.** Let $\mathcal{E}$ be an equation and $\tau\colon \tilde{\mathcal{E}} \to \mathcal{E}$ be a covering. By definition, a *τ-shadow* φ is a solution of the linearized equation $\tilde{\ell}_{\mathcal{E}}(\varphi) = 0$. The shadow fields $\tilde{\ni}_\varphi \in \bar{\mathrm{D}}_{\mathcal{C}}(\tilde{\mathcal{E}})$ are not true nonlocal symmetries since they do not describe the evolution of the nonlocal variable and, generally, not all of them can be extended up to nonlocal symmetries.

We have proved that Bäcklund transformations (161), (175), and (176) themselves do contain certain information about nonlocal variables such that these transformations can be integrated successfully. The corresponding non-abelian coverings in (178) provide nonlocal conservation laws for the underlying differential equations. Still, the structures on the covering equations are "too close" to the initial ones, so that the point symmetries of the initial equations and the classical symmetries of the covering equations are in one-to-one correspondence such that we get no nonlocal symmetries except the liftings of local transformations.

Now we introduce new nonlocal variables such that the symmetries we are in search of depend on them. Let $\overset{\leftrightarrow}{\Sigma}_t = \Xi_t + u_y$ be the new nonlocal variable such that

$$\begin{aligned}
&\tilde{D}_x^{\mathcal{E}_u}(\overset{\leftrightarrow}{\Sigma}_t) = 0,\\
&\tilde{D}_y^{\mathcal{E}_u}(\overset{\leftrightarrow}{\Sigma}_t) = \left(\overset{\leftrightarrow}{\Sigma}_t\right)^2 + u_{yy} - u_y^2 - \exp(2t).
\end{aligned}$$

Consider the limit of $\overset{\leftrightarrow}{\Sigma}_t$ as $t \to -\infty$. We see that at the point $t = -\infty$ there appeares the automodel variable

$$\Sigma_\infty = u_x + \frac{\exp(2u)}{\Xi_\infty}$$

such that $\tilde{D}_y^{\mathcal{E}_u}(\Sigma_\infty) = 0$. We claim that

$$\overset{\leftrightarrow}{\Sigma}_\infty = \lim_{t\to-\infty} \overset{\leftrightarrow}{\Sigma}_t$$

and Σ_∞ differ by the discrete symmetry $x \leftrightarrow y$. Indeed, consider the derivatives $\tilde{D}_x$ and $\tilde{D}_y$ of Σ_∞ and $\overset{\leftrightarrow}{\Sigma}_\infty$. We get

$$\tilde{D}_x(\Sigma_\infty) = \Sigma_\infty^2 + u_{xx} - u_x^2, \quad \tilde{D}_y(\Sigma_\infty) = 0.$$

Also, we have

$$\tilde{D}_x(\overset{\leftrightarrow}{\Sigma}_\infty) = 0, \quad \tilde{D}_y(\overset{\leftrightarrow}{\Sigma}_t) = (\overset{\leftrightarrow}{\Sigma}_\infty)^2 + u_{yy} - u_y^2.$$

Therefore we use the nonlocal variable Σ_∞ only and treat all relations up to the symmetry transformation $x \leftrightarrow y$ for the Liouville equation. By definition, put $\Sigma_t = (x \leftrightarrow y)\cdot(\overset{\leftrightarrow}{\Sigma}_t)$: we have

$$\tilde{D}_x(\Sigma_t) = \Sigma_t^2 + u_{xx} - u_x^2 - \exp(2t) \text{ and } \tilde{D}_y(\Sigma_t) = 0.$$

Nonlocal variables enable us to find shadows of nonlocal symmetries for Eq. (76) and to extend these shadows up to true nonlocal symmetries of the Liouville equation.

Proposition 69. (1) *Let $f(t,x,\Sigma_t)$ be a smooth function. Then the generating function*

$$\varphi=\frac{1}{2}({\Sigma_t}^2+u_{xx}-u_x^2-\exp(2t))\cdot\frac{\partial f}{\partial\Sigma_t}+\frac{1}{2}\frac{\partial f}{\partial x}+u_x\cdot f \tag{181}$$

is a τ_t-shadow of a nonlocal symmetry of the Liouville equation.

(2) *Let $f(x,\Sigma_\infty)$ be a smooth function. Then the second order τ-shadow $\varphi(x,\Sigma_\infty,u,u_x,u_{xx})$ for the Liouville equation is*

$$\varphi=\frac{1}{2}(\Sigma_\infty^2+u_{xx}-u_x^2)\cdot\frac{\partial f}{\partial\Sigma_\infty}+\frac{1}{2}\frac{\partial f}{\partial x}+u_xf=\tilde{\square}\big(f(x,\Sigma_\infty)\big). \tag{182}$$

Nonlocal shadows (181) *and* (182) *belong to the class* (*this class was considered in the paper* [99]) *of solutions* (65) *to the equation $\tilde{\ell}_F(\varphi)=0$. This class is now provided by the operator*

$$\tilde{\square}=u_x+\tfrac{1}{2}\tilde{D}_x$$

that contains the extended total derivative $\tilde{D}_x$.

Recontruction of nonlocal symmetries. In order to extend the τ_t-shadows $\tilde{\mathfrak{Э}}_\varphi$ up to true nonlocal symmetries

$$\tilde{\mathfrak{Э}}_{\varphi,a}=\tilde{\mathfrak{Э}}_\varphi+a\cdot\frac{\partial}{\partial\Sigma_t},$$

where $a\in C^\infty(\tilde{\mathcal{E}})$ and $t\in\mathbb{R}\cup\{-\infty\}$, we solve the equations

$$\tilde{D}_x(a)=\tilde{\mathfrak{Э}}_{\varphi,a}(\tilde{D}_x(\Sigma_t)),\quad \tilde{D}_y(a)=\tilde{\mathfrak{Э}}_{\varphi,a}(\tilde{D}_y(\Sigma_t))$$

for the function a.

Proposition 70. (1) *Let $f(t)$ be a smooth function and the functions φ and $a(t,\Sigma_t,u_x,u_{xx})$ be defined by the relations*

$$\varphi=u_x\cdot f(t),\quad a=(\Sigma_t^2+u_{xx}-u_x^2-\exp(2t))\cdot f(t). \tag{183}$$

Then the field $\tilde{\mathfrak{Э}}_\varphi+a\cdot\partial/\partial\Sigma_t$ is a true nonlocal symmetry of Eq. (76).

(2) *Let $f(x)$ be a smooth function and the functions φ and $a(\Sigma_\infty,$ u_x, $u_{xx})$ be given in*

$$\varphi=u_x\,f(x)+\frac{1}{2}\frac{df}{dx},\ a=(\Sigma_\infty^2+u_{xx}-u_x^2)\,f(x)+\Sigma_\infty\frac{df}{dx}+\frac{1}{2}\frac{d^2f}{dx^2}. \tag{184}$$

Then the field $\tilde{\mathfrak{Э}}_\varphi+a\cdot\partial/\partial\Sigma_\infty$ is a true nonlocal symmetry of Eq. (76).

The proof of Propositions 69 and 70 is very extensive and cannot be completed without application of the Jet ([72]) software for analytic transformations. The Jet environment allows to set the determining equations in a dialogue mode, select the simplest differential consequences to these equations, and then specify the expressions for the unknown symmetries.

Nonlocal symmetry (183) is defined up to elements of $\mathrm{CD}(\tilde{\mathcal{E}}) \ni g\cdot\tilde{D}_x$, $g \in C^\infty(\tilde{\mathcal{E}})$. Thence, the nonlocal symmetry class (183) is $[\tilde{\mathbf{\ni}}_{\varphi,a}] = [-f(t)\cdot\partial/\partial x]$, where $f(t)\cdot\partial/\partial x$ is the translation. Symmetry (184) is the lifting of a classical point symmetry φ_0^f, see Proposition 11 on page 38. As usual, a class of nonlocal symmetries $\tilde{\mathbf{\ni}}_{\varphi,a}$ can be obtained from Eq. (183) and (184) by using the discrete transformation $x \leftrightarrow y$.

11.3. **On permutability of Bäcklund transformations.** Now we illustrate the permutability property of Bäcklund (auto)transformations (161), (175), and (176).

Proposition 71. (1) *Let u^j, $j = \mathrm{i},\mathrm{ii}$, be solutions to Eq. (76) such that $\mathcal{B}_u(u,u^j;t_j) = 0$, $t_j \in \mathbb{R}$. Then there is a unique solution $u'''(x,y)$ to the system*

$$\left\{\begin{array}{rcl} \mathcal{B}_u(u',u''';t_2) &=& 0,\\ \mathcal{B}_u(u'',u''';t_1) &=& 0. \end{array}\right. \tag{185}$$

Namely, the solution u''' satisfies the relation

$$\exp(u''') = \exp(u)\cdot\frac{k_2\exp(u') - k_1\exp(u'')}{k_2\exp(u'') - k_1\exp(u')}, \tag{186}$$

where $k_j \equiv \exp(t_j)$.

(2) *Let $j = \mathrm{i},\mathrm{ii}$ and $t_j \in \mathbb{R}$. Suppose both v^j are solutions to the wave equation $v_{xy} = 0$ such that $\mathcal{B}_{uv}(u,v^j;t_j) = 0$. Also, suppose that u^j are solutions to the Liouville equation such that $\mathcal{B}_{uv}(u^j,v;t_j) = 0$. Then there are unique solutions u''' and v''' to the systems*

$$\left\{\begin{array}{rcl} \mathcal{B}_{uv}(u''',v';t_2) &=& 0\\ \mathcal{B}_{uv}(u''',v'';t_1) &=& 0 \end{array}\right. \quad \textit{and} \quad \left\{\begin{array}{rcl} \mathcal{B}_{uv}(u',v''';t_2) &=& 0\\ \mathcal{B}_{uv}(u'',v''';t_1) &=& 0, \end{array}\right.$$

respectively. Denote $k_j \equiv \exp(t_j)$, then the following relations hold:

$$\exp(u''') = \exp(u)\cdot\frac{k_2\exp(v') - k_1\exp(v'')}{k_2\exp(v'') - k_1\exp(v')},$$
$$\exp(v''') = \exp(v)\cdot\frac{k_1\exp(u'') - k_2\exp(u')}{k_2\exp(u'') - k_1\exp(u')}.$$

(3) *Let $j = \mathrm{i},\mathrm{ii}$ and $t_j \in \mathbb{R}$. Assume that Υ^j are solutions to the scal^+-equation $\mathcal{E}_\Upsilon$ such that $\mathcal{B}_{u\Upsilon}(u,\Upsilon^j;t_j) = 0$, and suppose that u^j are solutions to the Liouville equation such that*

$\mathcal{B}_{u\Upsilon}(u^j,\Upsilon;t_j)=0$. *Then there are unique solutions* u''' *and* Υ''' *to the systems*

$$\begin{cases}\mathcal{B}_{u\Upsilon}(u''',\Upsilon';t_2) &= 0\\ \mathcal{B}_{u\Upsilon}(u''',\Upsilon'';t_1) &= 0\end{cases}\qquad \textit{and} \qquad \begin{cases}\mathcal{B}_{u\Upsilon}(u',\Upsilon''';t_2) &= 0\\ \mathcal{B}_{u\Upsilon}(u'',\Upsilon''';t_1) &= 0,\end{cases}$$

respectively. We also have

$$\exp(u''')=\exp(u)\cdot\frac{k_2\exp(\Upsilon')-k_1\exp(\Upsilon'')}{k_2\exp(\Upsilon'')-k_1\exp(\Upsilon')},$$

$$\exp(\Upsilon''')=\exp(\Upsilon)\cdot\frac{k_1\exp(u'')-k_2\exp(u')}{k_2\exp(u'')-k_1\exp(u')},$$

where $k_j\equiv\exp(t_j)$.

Proof. We consider the case of Bäcklund autotransformation (161) only; cases 2 and 3 are treated analogously. Consider the subsystem in Eq. (185) that consists of relations (161) which contain the derivatives with respect to x only. Then the solution u''' defined in Eq. (186) follows from the linear dependence of the left-hand side within Eq. (185) and is unique. Consider another subsystem composed by the relations in Eq. (161) that contain the derivatives with respect to y. Then there are two solutions to this subsystem; we denote them by u''' and $\bar{u}'''$. The solution u''' is defined in Eq. (186) and $\bar{u}'''$ is obtained from the relation

$$\exp(\bar{u}''')=\exp(-u)\cdot\frac{k_1\exp(u')-k_2\exp(u'')}{k_2\exp(-u')-k_1\exp(-u'')}.$$

The solution $\bar{u}'''$ is irrelevant. Therefore, the function u''' is a unique solution to the whole system (185). ☐

Remark 28. Proposition 71 means that the diagrams

$$\begin{array}{ccc} u & \xrightarrow{t_1} & u' \\ {\scriptstyle t_2}\downarrow & & \downarrow{\scriptstyle t_2} \\ u'' & \xrightarrow[t_1]{} & u''' \end{array}\quad,\qquad \begin{array}{ccccc} u & \xrightarrow{t_1} & v' & \xrightarrow{t_3} & u'' \\ {\scriptstyle t_2}\downarrow & & \downarrow{\scriptstyle t_2} & & \downarrow{\scriptstyle t_2} \\ v'' & \xrightarrow[t_1]{} & u' & \xrightarrow[t_3]{} & v''' \end{array}\quad,$$

and the diagram

$$\begin{array}{ccccc} u & \xrightarrow{t_1} & \Upsilon' & \xrightarrow{t_3} & u'' \\ {\scriptstyle t_2}\downarrow & & \downarrow{\scriptstyle t_2} & & \downarrow{\scriptstyle t_2} \\ \Upsilon'' & \xrightarrow[t_1]{} & u' & \xrightarrow[t_3]{} & \Upsilon''' \end{array}$$

are commutative for any t_1, t_2, $t_3\in\mathbb{R}$.

12. Zero–curvature representations

In this section, we illustrate the relationship between the parametric families of zero–curvature representations and Bäcklund transformations for Eq. (158). In Sec. 1 we assigned the Liouville equation to the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, and now we benefit from the use of different representations of this algebra.

There is a natural equivalence ([13]) between $\mathfrak{g}$-valued zero–curvature representations of a PDE $\mathcal{E}$ and the special type coverings over the equation $\mathcal{E}$. Further on, we study the case $\mathfrak{g} \simeq \mathfrak{sl}_2(\mathbb{C})$. It is essential that there is a representation of $\mathfrak{sl}_2(\mathbb{C})$ in vector fields. We use it in order to construct the required coverings over the hyperbolic Liouville equation

$$\mathcal{E}_{\text{Liou}} = \{u_{xy} = \exp(2u)\}. \tag{187}$$

Here '2' is the Cartan 1×1-matrix of the Lie algebra A_1 while x and y are the coordinates in the standard two–dimensional extension $(z, \bar{z}) \hookrightarrow \mathbb{C}^2 \ni (x, y)$. By $\langle e, h, f\rangle$ we denote the canonical basis such that

$$[h, e] = 2e, \qquad [h, f] = -2f, \qquad [e, f] = h. \tag{50$'$}$$

Consider the representation $\rho\colon \mathfrak{sl}_2(\mathbb{C}) \to \mathrm{D}(\mathbb{C}_2[\Xi])$ of the Lie algebra $\mathfrak{g}$ in the space of polynomial-valued derivations:

$$\rho(e) = 1 \cdot \frac{\partial}{\partial \Xi}, \qquad \rho(h) = -2\Xi \cdot \frac{\partial}{\partial \Xi}, \qquad \rho(f) = -\Xi^2 \cdot \frac{\partial}{\partial \Xi} \tag{188a}$$

such that the Lie bracket is the commutator of vector fields: $[A, B] = A \circ B - B \circ A$. This representation of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ was used in the paper [52], where a class of N-ary analogs for the Lie algebraswas constructed. Also, consider the matrix representation

$$\varrho(e) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \varrho(h) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \varrho(f) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{188b}$$

such that the Lie bracket is the matrix commutator: $[A, B] = A \cdot B - B \cdot A$.

Given an equation $\mathcal{E}$, consider the flat connection form (47) in the bundle $C^\infty(\mathcal{E}^\infty) \otimes G \to \mathbb{C}^2$, where G is the Lie group of $\mathfrak{g}$. Suppose A, $B \in C^\infty(\mathcal{E}^\infty) \otimes \mathfrak{g}$. The zero-curvature condition (48), which is equivalent to the relation

$$[\bar{D}_x + A, \bar{D}_y + B] = 0,$$

is satisfied on the differential equation $\mathcal{E}$. Finally, we obtain the matrix equation

$$\bar{D}_y A - \bar{D}_x B - [A, B] = 0. \tag{189}$$

Now, decompose the matrices A and B with respect to the basis in the representation $\varrho\colon \mathfrak{g} \to \{M \in \mathrm{Mat}(2, 2) \mid \mathrm{tr}\, M = 0\}$:

$$A = a_e \otimes \varrho(e) + a_h \otimes \varrho(h) + a_f \otimes \varrho(f), \quad B = b_e \otimes \varrho(e) + b_h \otimes \varrho(h) + b_f \otimes \varrho(f),$$

where a_μ, $b_\nu \in C^\infty(\mathcal{E}^\infty)$, and construct the one-dimensional covering τ over $\mathcal{E}$ such that Ξ is the nonlocal variable, the extended total derivatives $\tilde{D}_x$ and $\tilde{D}_y$ are

$$\tilde{D}_x = \bar{D}_x + a_e \otimes \rho(e) + a_h \otimes \rho(h) + a_f \otimes \rho(f),$$
$$\tilde{D}_y = \bar{D}_y + b_e \otimes \rho(e) + b_h \otimes \rho(h) + b_f \otimes \rho(f),$$

respectively, and the rules to differentiate the variable Ξ are

$$\begin{aligned} \tilde{D}_x(\Xi) &= \mathrm{d}x \lrcorner \left(a_e \otimes \rho(e) + a_h \otimes \rho(h) + a_f \otimes \rho(f)\right), \\ \tilde{D}_y(\Xi) &= \mathrm{d}y \lrcorner \left(b_e \otimes \rho(e) + b_h \otimes \rho(h) + b_f \otimes \rho(f)\right). \end{aligned} \tag{190}$$

We see that the Maurer–Cartan condition (48), which is satisfied on $\mathcal{E}$, is equivalent to the compatibility condition $[\tilde{D}_x, \tilde{D}_y] = 0$ that holds in virtue of the equation $\mathcal{E}^\infty$.

Example 22. First, we obtain Bäcklund transformation between the Liouville and the wave equations. Consider Eq. (53) on page 34 and choose the gauge

$$a_e \equiv a_e^1 = \exp(\kappa u),\ \ b_f \equiv b_f^1 = \exp\big((2-\kappa)u\big),$$

for an arbitrary constant κ. Then the covering equation $\tilde{\mathcal{E}}$ is

$$\left\{ \begin{array}{l} v_x = (\kappa - 2)u_x + \exp(\kappa u - v) \\ v_y = \kappa u_y - \exp((2-\kappa)u + v) \end{array} \right\}. \tag{191}$$

Here the variable $v = \log \Xi$ is a transformation of the nonlocal variable Ξ, see Eq. (190). The compatibility condition for Eq. (191) is

$$v_{xy} = (\kappa - 1)\exp(2u).$$

Finally, suppose $\kappa = 1$, then Eq. (191) is the Bäcklund transformation ([14])

$$\begin{aligned} (v+u)_x &= \exp(u - v), \\ (v-u)_y &= -\exp(u + v) \end{aligned} \tag{191_1}$$

between the Liouville equation Eq. (187) and the wave equation

$$v_{xy} = 0, \tag{192}$$

while the coordinate Ξ is exactly the one that allows integrating equation (191_1) in nonlocal variables, see Sec. 11.

Remark 29. Transformation (191_1) is the particular case $t = 0$, $k \equiv \exp(t) = 1$ within the family of Bäcklund transformations (175) between Eq. (187) and Eq. (192). We note that the mapping $k \mapsto -k$ is the replacing of representation (188a) with the representation $\bar{\rho}\colon \mathfrak{sl}_2(\mathbb{C}) \to \mathrm{D}(\mathbb{C}_2[\Xi])$ such that

$$\bar{\rho}(e) = -1, \qquad \bar{\rho}(h) = -2\Xi, \qquad \bar{\rho}(f) = \Xi^2.$$

Zero-curvature representations constructed by using Bäcklund transformations. Consider Bäcklund autotransformation (161) for the Liouville equation (187) and Bäcklund transformation (176) between the Liouville equation and the Liouville scal$^+$-equation

$$\mathcal{E}_\Upsilon = \{\Upsilon_{xy} = \exp(-2\Upsilon)\}$$

(we recall that the latter equation is the hyperbolic representation of the Gauss equation for the conformal metric of constant curvature +1, see Example 10 on page 29). Next, consider the coverings over the Liouville equation that provide these transformations. Then, these coverings *exceed* ansatz (51).

Namely, the flat connection form for Bäcklund autotransformation (161) is

$$\theta = \begin{pmatrix} -\frac{1}{2}u_x & 0 \\ -\exp(u-t) & \frac{1}{2}u_x \end{pmatrix} \mathrm{d}x + \begin{pmatrix} \frac{1}{2}u_y & -\exp(t+u) \\ -\exp(t-u) & -\frac{1}{2}u_y \end{pmatrix} \mathrm{d}y.$$

The $\mathfrak{sl}_2$-valued flat connection form θ for Bäcklund transformation (176) is

$$\theta = \begin{pmatrix} -\frac{1}{2}u_x & \exp(-t-u) \\ -\exp(u-t) & \frac{1}{2}u_x \end{pmatrix} \mathrm{d}x + \begin{pmatrix} \frac{1}{2}u_y & -\exp(t+u) \\ 0 & -\frac{1}{2}u_y \end{pmatrix} \mathrm{d}y.$$

By construction, these $\mathfrak{sl}_2$-valued forms are zero–curvature representations for the Liouville equation.

Bäcklund transformations constructed by using zero–curvature representations. The problem of constructing multi–parametric families of Bäcklund transformations by using known zero–curvature representations for Eq. (187) was discussed in the paper [86]; later, this problem was studied thoroughly by V. Golovko in [32].

Now the exposition follows [86] and [32]. Let us find three $\mathfrak{sl}_2$-valued classes of zero–curvature representations $\theta \in \bar{\Lambda}^1(\mathcal{E}^\infty_{\text{Liou}}) \otimes \mathfrak{sl}_2(\mathbb{C})$ for the hyperbolic Liouvile equation (187). Assume that $A = A(u_x)$, $B = B(u)$, and suppose $[A, B] \neq 0$. Then Eq. (49′) on page 112 is reduced to

$$u_x^{-1}\frac{\partial A}{\partial u_x} - \exp(-2u)\frac{\partial B}{\partial u} - [u_x^{-1}A, \exp(-2u)B] = 0.$$

Applying $\partial^2/\partial u\,\partial u_x$ to this identity, we get the equation $[M,\, N] = 0$, where

$$M = \frac{\partial(A/u_x)}{\partial u_x}, \qquad N = \frac{\partial(B/\exp(2u))}{\partial u}.$$

There are three possible cases:

(1) $M = 0$,
(2) $N = 0$, and
(3) $M = r(u_x)\cdot C$, $N = s(u)\cdot C$, where $C \neq 0$ is a constant $\mathfrak{sl}_2(\mathbb{R})$-valued matrix nd r, $s \in C^\infty(\mathbb{R})$ are smooth functions.

Hence we get three gauge non-equivalent clases of zero–curvature representations:

Case 1 ($M = 0$). There are no nontrivial solutions to Eq. ($49'$).

Case 2 ($N = 0$). There are two classes of zero–curvature representations. They are

$$A = \begin{pmatrix} 2\alpha u_x + 2\beta & 2\alpha \\ u_x^2(1-2\alpha) - 4\beta u_x + 2\gamma & -2\alpha u_x - 2\beta \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 \\ \exp(2u) & 0 \end{pmatrix}, \tag{193a}$$

$$A = \begin{pmatrix} \alpha u_x^2 + 2\beta & 2\delta \exp(-2\alpha u_x) \\ 2\gamma \exp(2\alpha u_x) & -2\alpha u_x^2 - 2\beta \end{pmatrix},$$

$$B = \begin{pmatrix} \alpha \exp(2u) & 0 \\ 0 & -\alpha \exp(2u) \end{pmatrix}, \tag{193b}$$

Case 3 ($M \neq 0$, $N \neq 0$). Another solution to Eq. ($49'$) is

$$A = \begin{pmatrix} u_x & 1 \\ 0 & -u_x \end{pmatrix}, \; B = \begin{pmatrix} 0 & \alpha \exp(-2u) \\ \exp(2u) & 0 \end{pmatrix}, \tag{194}$$

where α, β, γ, and δ are arbitrary constants.

By [13], any $\mathfrak{sl}_2$-valued zero–curvature representation for $\mathcal{E}$ provides some special type covering over the equation $\mathcal{E}$. Consider the representation $\rho' \colon \mathfrak{sl}_2(\mathbb{C}) \to \mathrm{D}(\mathbb{C}[[v]])$:

$$\rho'(e) = \exp(-v)\frac{\partial}{\partial v}, \qquad \rho'(f) = -\exp(v)\frac{\partial}{\partial v}, \qquad \rho'(h) = -2\frac{\partial}{\partial v},$$

of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ in the space of differential operators on $\mathbb{C}$, *i.e.*, $v \in \mathbb{C}$ is the nonlocal variable and $\rho' \colon \mathfrak{sl}_2(\mathbb{C}) \to \mathrm{D}(\mathbb{C}[[v]])$. Then we construct one-dimensional coverings over the Liouvile equation $\mathcal{E}_{\mathrm{Liou}}$ that provide Bäcklund transformations between $\mathcal{E}_{\mathrm{Liou}}$ and some differential equations that depend on the initial zero–curvature representation.

Proposition 72 ([32])**.** *Representations* (193a)*,* (193b)*, and* (194) *correspond to Bäcklund transformations between* $\mathcal{E}_{\mathrm{Liou}}$ *and the equations*

$$\begin{aligned} v'_x = {} & \tfrac{1}{4}(2\alpha - 1)\exp(v')\left(\frac{v'_{xy}}{v'_y} - v'_x\right)^2 + \\ & 2(\beta\exp(v') - \alpha)\left(\frac{v'_{xy}}{v'_y} - v'_x\right) + 2\alpha\exp(-v') - 4\beta - 2\gamma\exp(v'), \end{aligned}$$

$$v''_x = -\frac{\alpha {v''_{xy}}^2}{2{v''_y}^2} - 4\beta + 2\delta\exp\left(-\frac{\alpha v''_{xy}}{v''_y} - v''\right) - 2\gamma\exp\left(\frac{\alpha v''_{xy}}{v''_y} + v''\right),$$

$${v'''_{xy}}^2 = \exp(-2v''')({v'''_y}^2 + 4\alpha).$$

If $\alpha = 0$ *in representations* (194)*, then we get Bäcklund transformation between* $\mathcal{E}_{\mathrm{Liou}}$ *and the wave equation Eq.* (192)*.*

Finally, we analyse the removability of the parameters α, β, γ, and δ in zero–curvature representations (193)–(194) with respect to the gauge transformations

$$A \mapsto SAS^{-1} - (D_x S)S^{-1}, \qquad B \mapsto SBS^{-1} - (D_y S)S^{-1}.$$

The result is described in

Remark 30 ([32]). The parameter β in Eq. (193a) is removable by the gauge transformation

$$S = a \cdot \begin{pmatrix} 1 & 0 \\ \beta/\alpha & 1 \end{pmatrix}$$

that depends on an arbitrary constant $a \in \mathbb{C}$; under this transformation $\gamma \mapsto \gamma + \beta^2/\alpha$. All other parameters α, β, γ, and δ in zero–curvature representations (193)–(194) are non-removable.

Appendix A. Geometric methods of solving boundary–value problems

In this appendix, we consider several methods for solving boundary–value problems (mainly, the Dirichlet problem) for nonlinear equations of mathematical physics. Then we compare and analyse the results of a computer experiment in applying the described algorithms, see also [49]. Again, the methods which we study are based on the geometry of the jet spaces ([10, 63, 81]) and treating differential equations $\mathcal{E}$ (and their prolongations $\mathcal{E}^\infty$ as well) as submanifolds $\mathcal{E} \subset J^k(\pi)$ of the jet space of order k for a certain fibre bundle π (respectively, we have $\mathcal{E}^\infty \subset J^\infty(\pi)$). By using this approach, we pay attention to the analysis of the following objects: they are

- differential equations and sections s such that the jet $j_k(s) \subset \mathcal{E}$ defines a solution of $\mathcal{E}$,
- boundary–value problems $\mathcal{P} = (\mathcal{E}, \mathcal{D}, f = s|_{\partial\mathcal{D}})$ in a domain $\mathcal{D}$,
- and their deformations $\dot{\mathcal{P}}$.

We emphasize that the geometrically motivated technology of solving the boundary–value problems is discussed within this appendix. We do not stop on particular properties of some solutions in practical situations. Basic definitions and concepts were formulated in the Introduction, see also [10, 63, 79, 81]. We use the elliptic Liouville equation

$$\mathcal{E}_{\text{Liou}} = \{u_{z\bar{z}} = \exp(2u)\}$$

as a basic example.

The appendix is organized as follows. In Sec. A.1, the method of monotonous iterations for the solutions u^t of the boundary–value problem $\mathcal{P}$ is described. Here $t \in \mathbb{N}$. This method is based of the theory of differential inequalities ([79]). In Sec. A.2, we construct a method for solving the boundary–value problems that involves the evolution representation of the equation $\mathcal{E}$ under study. We shall see that this interpretation is admissible in a sufficiently general situation ([10, 81]). Then, in Sec. A.3 we consider various methods based on the deformations $\dot{f}$ of the boundary conditions f and simultaneous invariance of the equation $\mathcal{E}$:

$$\dot{\mathcal{P}} = (\mathcal{E}, \mathcal{D}, \dot{f}).$$

In Sec. A.4 we describe the relaxation method based on the substitution

$$\mathcal{P} \mapsto \mathcal{P}' = (\mathcal{E}', \mathcal{D} \times \mathbb{R}_+, f \otimes \mathrm{id}_t)$$

of the boundary–value problem such that solutions u of the problem $\mathcal{P}$ are stable stationary solutions for the new problem $\mathcal{P}'$ with respect to the evolution equation $\mathcal{E}'$. The deformations

$$\dot{\mathcal{P}} = (\dot{\mathcal{E}}, \mathcal{D}, f),$$

which are considered in Sec. A.5, are in some sence antipodal to ones described in Sec. A.3. Now, the boundary condition f is invariant and the equation $\mathcal{E}(t)$ moves. The "planting" of a nonlinearity could be an example. In the final section of this appendix, we discuss the results of a computer experiment in practical application of all these methods.

Remark 31. In what follows, we assume that there is a unique classical solution to the boundary–value problem at hand (see [34]). Also, we recall that the transformation laws of solutions with respect to transformations of the independent variables are known for some equations, *e.g.*, the conformally invariant Toda equations, see Proposition 11 on page 38). Therefore, the domain $\mathcal{D}$, where the problem $\mathcal{P}$ is solved, can be chosen relatively regular (we set $\mathcal{D} \sim B_0^1$ for Eq. (55)).

Remark 32. First, we point out a practically useful way to construct a set of solutions (that can be large enough) of a boundary–value problem $\mathcal{P} = (\mathcal{E},\, \mathcal{D},\, f)$. Assume that the equation $\mathcal{E}$ is invariant with respect to a vector field X. Denote its generating section by φ_X and suppose that A_t is the flow of the field X such that

$$A_t(\partial\mathcal{D}) = \partial\mathcal{D} \text{ and } A_t(j_k(f)) = j_k(f),$$

that is, the boundary condition is unvariant with respect to the symmetry X of the equation $\mathcal{E}$ at hand. Then problem $\mathcal{P}$ can be reduced to finding solutions of the boundary–value problem

$$\mathcal{P}' = (\mathcal{E} \cap \{\varphi_X = 0\}, \mathcal{D}, f)$$

that are invariant with respect to X at any point in $\mathcal{D}$. If the problem $\mathcal{P}'$ has a solution, then, in general, this solution may be not unique.

Example 23. Consider the Dirichlet boundary–value problem

$$\bigl\{u_{xx} + u_{yy} = \exp(2u), \qquad u\big|_{r=1} = f, \quad f \in C^{0,\lambda}\bigr\} \tag{195}$$

in the unit disc B_0^1 and assume that the boundary condition is homogeneous and trivial, $f \equiv 0$. The solutions for the Liouville equation $\mathcal{E}_{\text{Liou}}$ that are invariant with respect to its point symmetries are

$$u = \frac{1}{2}\log\frac{v_x^2 + v_y^2}{\sinh^2 v}, \tag{196a}$$

$$u = \frac{1}{2}\log\frac{v_x^2 + v_y^2}{v^2}, \tag{196b}$$

$$u = \frac{1}{2}\log\frac{v_x^2 + v_y^2}{\sin^2 v}, \tag{196c}$$

where v is a harmonic function,

$$\Delta v = 0.$$

Consider the radial symmetry

$$X = -y\,\frac{\partial}{\partial x} + x\,\frac{\partial}{\partial y}.$$

Then the solutions $u(r) = u(\sqrt{x^2+y^2})$ of problem (195) are the following:

$$u_1(r) = \log \frac{2(1+\sqrt{2})}{(1+\sqrt{2})^2 - r^2}, \tag{197a}$$

$$u_2(r) = -\log r - \log(1 - \log r), \tag{197b}$$

$$u_3(r) = \frac{1}{2} \log \frac{\alpha^2 r^{-2}}{\sin^2(\alpha\beta - \alpha \log r)}, \qquad \beta = \frac{\pm 1}{\alpha} \arcsin \alpha,\ \alpha \neq 0. \tag{197c}$$

A unique classical solution u_1 has no singularity at the point 0. The solution u_2 has an integrable singularity at $r = 0$, and its gradient on the border vanishes. The solution u_3 has the cardinal set of logarithmic singularities along the radius r. These singularities accumulate as $r \to 0$.

Remark 33. Suppose that the equation $\mathcal{E}$ at hand is Euler. Then, obviously, one can search solutions of the Dirichlet boundary–value problem by using the direct minimization of the action functional. We emphasize that the projective methods can be used in addition to the lattice discretization methods.

A.1. **The monotonous iterations method.** The monotonous iterations method is a useful instrument in the theory of differential inequalities ([79]). This method allows to contruct solutions of the Dirichlet boundary–value problems

$$\big\{\Delta u = h(u, x), \quad u|_{\partial\mathcal{D}} = f, \qquad \partial\mathcal{D} \in C^{1+\varepsilon}, \quad f, h \in C^{0,\lambda}\big\} \tag{198}$$

and then check the local uniqueness of these solutions.

Definition 13. A function $\alpha \in C(\overline{\mathcal{D}}) \cap C^2(\mathcal{D})$ (resp, β) is a *lower* (*upper*) solution of problem (198) is the following two conditions hold:

(1) $\Delta\alpha - h(\alpha, x) \geq 0$ in $\mathcal{D}$ and
(2) $f(x) \geq \alpha(x)$ on $\partial\mathcal{D}$

(respectively, $\leq$). Suppose further $\alpha(x) \leq \beta(x)$ for each $x \in \overline{\mathcal{D}}$. Then we set $\alpha \preceq \beta$.

Proposition 73 ([79])**.** *Assume that a lower and an upper solutions of problem* (198) *are ordered, $\alpha \preceq \beta$. Suppose there is a nonnegative constant $C \in \mathbb{R}$ such that*

$$h(u_1, x) - h(u_2, x) \leq C \cdot (u_1 - u_2)$$

for any $x \in \overline{\mathcal{D}}$, u_1, and u_2 provided that $\alpha \leq u_2 \leq u_1 \leq \beta$. Consider two sequences $\underline{U} = \{\underline{u}^k\}$ and $\overline{U} = \{\overline{u}^k\}$ of solutions of the problem

$$\Delta u^k - C\, u^k = h(u^{k-1}, x) - C\, u^{k-1}, \quad u^k|_{\partial\mathcal{D}} = f, \qquad x \in \mathcal{D},\ k \in \mathbb{N}. \tag{199}$$

Here the inital values are $\underline{u}^0 = \alpha$ and $\overline{u}^0 = \beta$, respectively. Then the monotonously nondecreasing (nonincreasing) sequence $\underline{U}$ ($\overline{U}$) converges

to a solution $\underline{u}$ *(resp.,* $\overline{u}$*) for problem* (198)*. Moreover, we have* $\underline{u} \le \overline{u}$ *in* $\overline{\mathcal{D}}$ *and the bound* $\underline{u} \le u_* \le \overline{u}$ *holds for any solution* $u_* \in [\alpha, \beta]$*.*

Suppose further that h *is monotonous with respect to* u*:* $h(u_1, x) - h(u_2, x) \ge 0$ *provided that* $\alpha \preceq u_2 \preceq u_1 \preceq \beta$*. Then the solution for problem* (198) *is unique on* $[\alpha, \beta]$*:* $\underline{u} = \overline{u}$*.*

The notions of a lower and upper solution, which were introduced in Sec. A.1, and the method of proving the local uniqueness for solutions of problem (198) will be used in Sec. A.4. The relaxation method will be described there. In that case, the evolution of the superscript k is continuous unlike in problems (199).

A.2. **Evolutionary representation of differential equations.** The following remarkable result was in fact obtained in the papers concerning the formal theory of differential equations (see [10, 81] and references therein): any equation that satisfies the assumtions of the '2-line theorem' for the Vinogradov's $\mathcal{C}$-spectral sequence ([10, 63]) admits a representation in the form of an evolution equation. An illustration is given in Example 3 on page 12. In practice, this means that any equation that does not have gauge symmetries (unlike the Maxwell, the Yang–Mills, and the Einstein equations and similar systems) admits a set of coordinate transformations and differential substitutions that map it to an evolution equation (possibly, the new equation is imposed on a larger set of dependent variables). The class of equations subject to the assumptions of the '2-line theorem' is really wide. Of course, fixed-precision integrating of evolution equations is simpler that integrating of arbitrarily chosen equations of mathematical physics. Therefore, solving the initial boundary–value problem is divided to two stages. First, we find an evolution representation of the equation $\mathcal{E}$. Then we reconstruct the boundary conditions for the auxiliary dependent variables if there appeares a necessity to introduce them.

Example 24. Consider the boundary–value problem

$$\begin{gathered} u_{xx} \pm u_{yy} = \exp(2u), \qquad \mathcal{D} = \{(x, y), |x| < 1, |y| < 1\}, \\ u(1, y) = f_1(y),\ u(x, 1) = f_2(x),\ u(-1, y) = f_3(y),\ u(x, -1) = f_4(x) \end{gathered} \tag{200}$$

for the elliptic (resp., hyperbolic) Liouville equation $\mathcal{E}_{\text{Liou}}$. Introduce the additional dependent variable $v = u_y$. Then the equation $\mathcal{E}_{\text{Liou}}$ has the form

$$\frac{\partial}{\partial y} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ \mp u_{xx} \pm \exp(2u) \end{pmatrix}$$

and the conditions on the bound $\partial\mathcal{D}$ are split to the initial and the boundary conditions:

$$\begin{aligned} &u(x, -1) = f_4(x), \quad &&u(1, y) = f_1(y), \quad &&u(-1, y) = f_3(y) \\ &v(x, -1) = v[f_2], \quad &&v(1, y) = df_1(y)/dy, \quad &&v(-1, y) = df_3(y)/dy. \end{aligned}$$

Thence, solving boundary–value problem (200) is reduced to reconstruction of the initial section $v(x,-1)$ by the terminal section $u(x,1) = f_2(x)$. In practice, this can be done by using the conjugated gradient method.

A.3. **Evolution of boundary conditions.** Recall that any symmetry $\varphi \in \operatorname{sym}\mathcal{E}^\infty$ of an equation $\mathcal{E} = \{\vec{F} = 0\}$ is an element of the kernel $\ker \bar{\ell}_F$ of the linearization

$$\ell_F = \left\| \sum_\sigma \frac{\partial F_i}{\partial u^j_\sigma} D_\sigma \cdot \mathbf{1}_{ij} \right\|$$

restricted onto $\mathcal{E}$. The evolutionary vector field

$$\ni_\varphi = \sum_{j,\sigma} \bar{D}_\sigma(\varphi^j)\, \frac{\partial}{\partial u^j_\sigma}$$

commutes with the total derivatives D_i and is tangent to the infinite prolongation

$$\mathcal{E}^\infty = \{D_\sigma(F) = 0,\ |\sigma| \geq 0\} \subset J^\infty(\pi).$$

The latter is the projective limit of smooth epimorphisms by definition. The manifold $\mathcal{E}^\infty$ that is defined by the closed algebra of smooth functions is in fact infinite–dimensional, therefore, formally, the field $\ni_\varphi$ has no flow. Indeed, there is no Cauchy theorem for the initial–value problem

$$\dot{u}_\sigma = \bar{D}_\sigma(\varphi), |\sigma| \geq 0, \qquad u(t=0) = u \tag{201}$$

composed by the cardinal set of equations. The integral trajectories for problem (201) exist provided that $\varphi(x, u, D_i(u^j))$ is a contact symmetry. In particular, the point summetries that are linear with respect to the derivatives suit well. From Eq. (201) it follows that the functions φ describe the correlated evolution of the dependent variables and their derivatives. This evolution preserves solutions of the equation $\mathcal{E}$ if $\varphi \in \ker \bar{\ell}_F$.

A.3.1. Suppose a solution u_0 of some boundary–value problem $\mathcal{P}_0 = (\mathcal{E},\ \mathcal{D},\ f_0)$ for an equation $\mathcal{E}$ is known. Then, deform the problem $\mathcal{P} = (\mathcal{E}, \mathcal{D}, f)$ such that $f(t=0) = f_0$ and $f(t=1) = f$. Here by $\dot{f}(t)$ we denote the deformation velocity of the boundary condition on $\partial\mathcal{D}$ and by $\varphi(t)$ the corresponding deformation of the solution. One easily checks that $\varphi \in \ker \bar{\ell}_F$. Suppose further that the problem

$$\varphi \in \ker \bar{\ell}_F, \qquad \varphi|_{\partial\mathcal{D}} = \dot{f} \tag{202}$$

is soluble at each $t \in [0,1)$. Then,

$$u = u_0 + \int_0^1 \varphi(t)\, dt$$

is the required solution for the problem $\mathcal{P}$.

The condition $\varphi \in \ker \bar{\ell}_F$ is the decomposition of the deformation $\dot{u} = \varphi$ to a power series in t. The analysis of the deformation equations that are coefficiets of the higher powers of t is interesting by itself, see [64]. For example, insolubility of the equation at t^2 implies impossibility to solve the problem $\mathcal{P}$ by using this deformation method.

Remark 34. The Liouville equation $\mathcal{E}_{\text{Liou}}$ has the following peculiar feature. Its general solution, see Eq. (87), that depends on an arbitrary holomorphic function $v(z)$ is always invariant with respect to a point symmetry X whose coefficients are related with $v(z)$ by the Abel transformation. The curve $X(t)$ in the space $\operatorname{sym} \mathcal{E}_{\text{Liou}}$ is assigned to the deformation $\dot{f}$ of the boundary condition. This curve is such that the solution $u(t')$ is invariant with respect to $X(t')$ for any $t' \in [0, 1]$. Therefore, the solution of boundary–value problem (195) can be reduced to analysis of the equations that define the curve $X(t)$.

A.3.2. Consider a particular covering strucuture $\tilde{\mathcal{E}}^\infty \to \mathcal{E}^\infty$ over the equation $\mathcal{E}$ at hand, that is, suppose that the substitution $u = u[v]$ maps solutions $v \in \operatorname{Sol} \tilde{\mathcal{E}}$ to solutions $u \in \operatorname{Sol} \mathcal{E}$. Several examples are well known: the Cole–Hopf substitution

$$u = \frac{v_x}{v}, \tag{203}$$

the Miura transformation $u = v_x - v^2$, and the formula

$$u = \frac{1}{2} \log[4\partial v \cdot \bar{\partial}\bar{v}/(1 - v\bar{v})^2] \tag{204}$$

by Liouville ([69]) that relate the Burgers and the heat equations, the Korteweg–de Vries equation and the modified Korteweg–de Vries equation, and the Liouville and the Cauchy–Riemann equations, respectively. We recall that all three expressions (196) are transformed to Eq. (204) by an appropriate change of variables. Also, suppose the condition f is fixed. Then there can be several substitutions $u[v]$ that solve the boundary–value problem $\mathcal{P}$. Therefore, the reconstruction problem for the function v in the whole domain $\mathcal{D}$ is incorrect by Hadamars. Still, assume that a class of the substitutions $v \mapsto u$ is fixed. Then one easily obtains the coordinate expressions for the equations

$$\psi \equiv \dot{v} \in \ker \tilde{\ell}_{\tilde{\mathcal{E}}},$$

which are analogous to Eq. (202), plus the condition

$$\tilde{\mathfrak{I}}_\psi(u[v]) = \dot{f}$$

defined by the boundary functions f and f_0, and the quadrature

$$v(1) = v(0) + \int_0^1 \psi(t)\, dt.$$

Thence we construct the solution u of the initial boundary–value problem $\mathcal{P}$ in the whole domain $\mathcal{D}$ by using the exact formula $u = u[v]$

and the solution $v(1)$ of the covering equation $\tilde{\mathcal{E}}$. Still, we see that the condition $\tilde{\ni}_\psi(u[v]) = \dot{f}$ may not even be defined by an operator with directional derivative if the function u depends on the gradient grad v explicitly.

Nawadays, there exist regular algorithmic methods ([10], see also Chapter 4) that allow obtaining and classification of the coverings over equations of mathematical physics. These algoriths are already available as environments ([72]) for the symbolic transformations software. We also recall that the concept of coverings over differential equations is closely related with the theory of Bäcklund transformations and zero–curvature representations (see the preceding section) for PDE. The latter structures also provide some classes of the substitutions $u[v]$ for a given equation $\mathcal{E}$.

A.3.3. Now we consider in more details solving the Dirichlet boundary–value problem, see Eq. (195), for the Liouville equation $\mathcal{E}_{\text{Liou}}$ within the class of substitutions (196b) by using the methods described above. Recall that the harmonic functions $v(t)$ admit the representations $P[g]$ via the Poisson kernel P (see [34]) by their boundary value $g \equiv v|_{\partial\mathcal{D}}$. Therefore, the initial boundary–value problem $\mathcal{P}$ which is solved by using the homotopy $\dot{f}$ of the solution u such that $f(t)$ are its values on $\partial\mathcal{D}$ is reduced to the integral equation

$$\frac{d}{dx}P[g]\cdot\frac{d}{dx}P[\dot{g}] + \frac{d}{dy}P[g]\cdot\frac{d}{dy}P[\dot{g}] = (g\dot{g} + g^2\cdot(f - f_0))\cdot\exp(2f)$$

with respect to the deformation $\dot{g} = \psi|_{\partial\mathcal{D}}$ of the bounder value for the harmonic function v. Suppose the boundary value for v is obtained, then an approximation of v on a lattice in $\mathcal{D}$ can be obtained by using multiple averaging of the values of v in the neighbouring interpolation points (see also Sec. A.4 below).

In Remark A.6.2 on page 125, the results of a computer experiment in application of this method for solving boundary–value problem (195) are discussed.

A.3.4. Finally, we note that solutions of some other equation $\mathcal{E}'$ (in general, distinct from the equation $\mathcal{E}$ at hand) can be used for construction of the initial sections u_0 that correspond to the boundary values f_0 on $\partial\mathcal{D}$. Here we assume that the solutions $u(t)$ satisfy a degenerate equation $\mathcal{E}'$ as $t \to 0$.

Example 25. Suppose $f \rightrightarrows -\infty$ such that $\max f - \min f \leq \text{const} < \infty$. Then a solution u of boundary–value problem (195) in the disc B_0^1 approximated with respect to the norm $\|\cdot\|_{C^0(\bar{B}_0^1)}$ by the harmonic function $P[f]$ as closely as desired. Let

$$\Delta u(t) = \exp(2u(t)), \qquad f(t) = f + \log t, \quad 0 < t \leq 1,$$

then in the same notation we have

$$\Delta\varphi(t) - 2\exp(2u(t))\varphi(t) = 0, \quad \dot{f}(t) = t^{-1}, \quad u(t) = u_{t_{\min}} + \int_{t_{\min}}^{t} \varphi(\tau)d\tau,$$

where $t_{\min} \to +0$ and $u_{t_{\min}} = P[f + \log t_{\min}]$ is the initial approximation. Now suppose t is small. We expand the solution $u(t)$ by using the Hadamars lemma and obtain $u(t) = \log t + P[f] + t^2 \cdot U$. If $t \to +0$, then the function U satisfies the homogeneous problem

$$\Delta U = \exp(2P[f]), \qquad U|_{\partial B_0^1} = 0$$

for the Poisson equation. Its solution is

$$U(p) = \int_{B_0^1} \exp(2P[f])(q) \cdot G(p,q)\,dq,$$

where G is the Green function ([34]) for the Laplace operator in the unit disc.

In fact, the method described in Sec. A.3.4 is based on the simultaneous deformation $\dot{\mathcal{P}}$ of the boundary condition f and the equation $\mathcal{E}$ itself. Indeed, we have $\mathcal{E}(0) = \mathcal{E}'$ and $\mathcal{E}(t) = \mathcal{E}$ for $t \in (0,1]$ by construction. In Sec. A.4, we consider an opposite situation: an auxiliary equation $\mathcal{E}'$ is used at any value of the parameter t. Here we assume that solutions of $\mathcal{E}'$ tend to solutions of the problem $\mathcal{P}$ as t increases.

A.4. **The relaxation method.** Complement the mixed boundary–value problem, which is a generalization of problem (198), by the relaxation term $\partial u/\partial t$. Assume that α and β are the lower and the upper solutions of the initial stationary problem, respectively. Then, extend the boundary value f onto $\partial\mathcal{D} \times \mathbb{R}_+$ and fix an initial approximation u_0 such that $\alpha \preceq u_0 \preceq \beta$: hence we obtain

$$\frac{\partial u}{\partial t} - \Delta u = -h(u,x), \quad a\,\frac{\partial u}{\partial \vec{n}} + b\,u = f, \quad u(x,0) = u_0(x). \tag{205}$$

Proposition 74 ([79])**.** *Let the above assumptions hold. Assume further that $h \in C_u^1[\alpha,\beta]$. Then the following two statements are equivalent*:

(1) *the stationary solution u_s of problem* (205) *is unique in* $[\alpha,\beta]$;
(2) *the solution $u_s \in [\alpha,\beta]$ is asymptotically stable such that the stability domain is* $[\alpha,\beta]$.

Here we discussed a method based on the replacement of the elliptic equation $\mathcal{E}$ in problem (198) by the evolution equation $\mathcal{E}'$. This method is an alternative to the evolutionary representation method for the initial equation $\mathcal{E}$ (see Sec. A.2).

A.5. **Deformation of the equation.** Now we consider the deformations $\dot{\mathcal{P}}$ of the boundary–value problem $\mathcal{P}(t)$ that are induced by the evolution $\dot{\mathcal{E}}$ of the equation $\mathcal{E}(t) = \{F(t) = 0\}$ at hand. Here we assume that the boundary value f remains invariant. Namely. consider the homotopy

$$\mathcal{E}(t) = (1 - r(t)) \cdot \mathcal{E}_0 + r(t) \cdot \mathcal{E},$$

where $r(0) = 0$ and $r(1) = 1$. Then the velocity φ of deformation of the solution $u(t)$ is subject to the boundary–value problem

$$\bar{\ell}_{F(t)}(\varphi) + r'(t) \cdot (F - F_0) = 0, \qquad \varphi|_{\partial\mathcal{D}} = 0.$$

Therefore, the solution of the problem $\mathcal{P}$ is defined by the quadrature $u = u_0 + \int_0^1 \varphi(t)\,dt$. We emphasize that this method provides reliable solution approximations in computations.

A.6. **Discussion and practical hints.**

A.6.1. Consider the deformation $(1-r(t))\cdot\mathcal{P}_0+r(t)\cdot\mathcal{P}$ of the boundary values (see Sec. A.3) or of the equation itself (see Sec. A.5). Then the smooth step-like homotopy function

$$r(t) = \exp(-\mathrm{ctg}^2(\pi t/2)), \qquad 0 < t < 1,$$

is more preferable than the linear motion $\dot{\mathcal{P}} = \mathcal{P} - \mathcal{P}_0$. Then, the final stage of solving the boundary–value problem $\mathcal{P}$ is in fact equivalent to the Newton method that starts with the approximation obtained in the initial computations.

A.6.2. The comparative analysis of practical computations by using the algorithms described in this appendix was carried out in 2002 in the dimploma papers by A. V. Punina (Chair of Higher Mathematics, ISPU; Master thesis "Comparative analysis of the methods for solving the elliptic Liouville equation by using the homotopies") and N. P. Cheluhoeva (Chair of Higher Mathematics, ISPU; Master thesis "Integral equations on the border method studied for solving the elliptic Liouville equation"). The following assertion is in order.

- The method of deforming the equation $\mathcal{E}$ (see Sec.A.5) is twice more precise with respect to the absolute deviation from known exact solutions than the method based on the deformation of the boundary function, see Sec. A.3, although the latter method converges 3-5 times faster.
- The relaxation method is the simplest among the algorithms such that the evolutions of t is continuous. The interative method is also preferable for solving the Laplace equation that appeared in Sec. A.3 since the use of the Poisson kernel requires greater time and the Gauss method cannot be speeded up owing to the strongly sparse matrix of the discrete Laplace operator Δ. In Sec. A.3, we described the reduction of the Dirichlet

border–value problem to the one–dimensional problem and the integral equation. This method is very sensitive with respect to smoothness of the boundary conditions $f(t)$ since the Poisson kernel has a singularity on ∂B_0^1 and the harmonic functions are less smooth on ∂B_0^1 than in $\mathcal{D}$ (where they are infinitely smooth). We also see that a solution of the Liouville equation, when moving along t, can be "attracted" to a nonclassical (of type (197b) or (197c)) solution which satisfies the same boundary conditions. This can happen if the time step is suffucuently large.

Summarizing, we conclude that all these methods demonstrated comparable precisions. Therefore, the choice of an algorithm should be based on the actual properties of the problem at hand.

A.6.3. Conservation laws for the analysed equation $\mathcal{E}$ serve an important instrument for the precision control in computations. Nowadays, there exist regular methods ([10, 63, 94]) of reconstruction of the exhaustive set of conservation laws for the equations subject to the '2-line theorem'. These methods are realized in the form of the computer software ([72]) for systems of analytic transformations.

A.6.4. Application of the methods of soving boundary–value problems described in this appendix does neither neglect nor underesteeme the standard check of continuity of the resulting solutions, their Hölder or the Sobolev space properties an so on. Meanwhile, we hope that these practical ideas will compliment the everyday set of instruments for numerical analysis of nonlinear equations of mathematical physics.

Final remarks

1. Recently, Demskoi and Startsev ([19]) analysed the correlation between the integrals $\mathbf{\Omega}$ and symmetries φ for the Liouvillean hyperbolic systems. They assigned the operators $\bar{\square}$ that factor symmetries of these systems to the product

$$\varphi = \bar{\square}(\phi(x, \mathbf{\Omega}))$$

to the linearizations ℓ_{Ω^i} of the integrals Ω. These results are reported in the note [19], which is found in this issue. In fact, they generalize the statements of Lemma 13 on page 39 and Lemma 50 on page 78 to the case of arbitrary integrals Ω^i, $i \geq 1$ (we recall that $\Omega^1 \equiv T$).

2. The commutative Hamiltonian hierarchy $\mathfrak{A}$ of the local Noether symmetries $\varphi_k \in \operatorname{sym} \mathcal{L}_{\text{Toda}}$, where $k \geq 0$, was constructed in Part I. We identified this hierarchy with the sequence of higher r-component analogs for the potential modified Korteweg–de Vries equation and related $\mathfrak{A}$ with the bi–Hamiltonian hierarchy $\mathfrak{B}$ for the scalar potential Korteweg–de Vries equation, see Eq. (93) on page 51. The degeneracy of the matrix coefficients of the higher-order derivtives is an immanent feature of these evolution equations.

A comment is in order. In the fundamental paper [22], the integrable bi–Hamiltonian hierarchy $\mathfrak{C}$ of the Drinfel'd–Sokolov equations was assigned to the hyperbolic Toda system associated with a semisimple Lie algebra $\mathfrak{g}$ or a Kac–Moody algebra $\hat{\mathfrak{g}}$. The hierarchy $\mathfrak{C}$ was related with the sequence $\mathfrak{D}$ of the multi–component Kortewegde Vries equation's analogs. Unlike in the case of the hierarchy $\mathfrak{A}$, the symbols of the Drinfel'd–Sokolov equations are always nondegenerate. Therefore, it is quite reasonable to check whether there is an interrelation between these two systems $\mathfrak{A}$ and $\mathfrak{C}$, as well as $\mathfrak{B}$ and $\mathfrak{D}$. Also, the necessary and sufficient conditions for $\mathfrak{A}$ to be bi–Hamiltonian remain unclear.

Suppose the hierarchy $\mathfrak{A}$ is assigned to the Toda equation (55) which is associated with a nondegenerate symmetrizable matrix K. We conjecture that the hierarchy $\mathfrak{A}$ is bi-Hamiltonian if and only if K is the Cartan matrix of a semisimple Lie algebra. We also conjecture that the pair (A_1, A_2) of the operators is Hamiltonian in the sence of Definition 7 on page 10 under the same assumptions.

3. Chapter 3 contains the exposition of the geometric structures for the scalar heavenly equation, which is a continuous dispersionless limit of the r-component Toda systems as r tends to infinity. We observed that local structures for the limit equation are relatively few, and therefore a nonlocal setting must be introduced. Meanwhile, local Noether's symmetries, conservation laws, and recursion operators for the Toda equations themselves were considered in Part I. Therefore it is logical

to describe the permutability properties of the diagram

$$\begin{array}{ccc}
\mathcal{E}_{\text{Toda}} & \longrightarrow & \text{local structures for } \mathcal{E}_{\text{Toda}} \\
{\scriptstyle r\to\infty,\ \varepsilon\to+0,\ u_{zzz}=0}\Big\downarrow & & \Big\downarrow{\scriptstyle ?} \\
\mathcal{E}_{\text{heav}} & \longrightarrow & \text{nonlocal structures for } \mathcal{E}_{\text{heav}},
\end{array}$$

that links together the local geometry of $\mathcal{E}_{\text{Toda}}$ and (yet undiscovered) nonlocal geometry of $\mathcal{E}_{\text{heav}}$.

4. In Chapter 4, we applied the cohomological schemes developed by I. S. Krasil'shchik and constructed one–parametric families of Bäcklund transformations for Eq. (158) only, leaving apart the general case ([3]) of Bäcklund transformations for the Toda equations associated with the semisimple Lie algebras. Still, from the resulting expressions in [3] it is clear that the scaling symmetry is the required generator of deformations for any $r \geq 1$ and any $\mathfrak{g}$ of rank r.

The author hopes that the reasonings of the present paper demonstrate the profits one obtains by using the invariant coordinate–free approach towards the mathematical physics equations. The reader will enjoy an excursion to the world of the PDE-related algebraic structures in the recent paper [52], where a natural class of N-ary generalizations for the Lie–algebra structures (in particular, of the symmetry algebra $\operatorname{sym}\mathcal{E}_{\text{Toda}}$ for the Toda equations) was considered.

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